An Algorithmic Approach to Stability Verification of Hybrid Systems: A Summary

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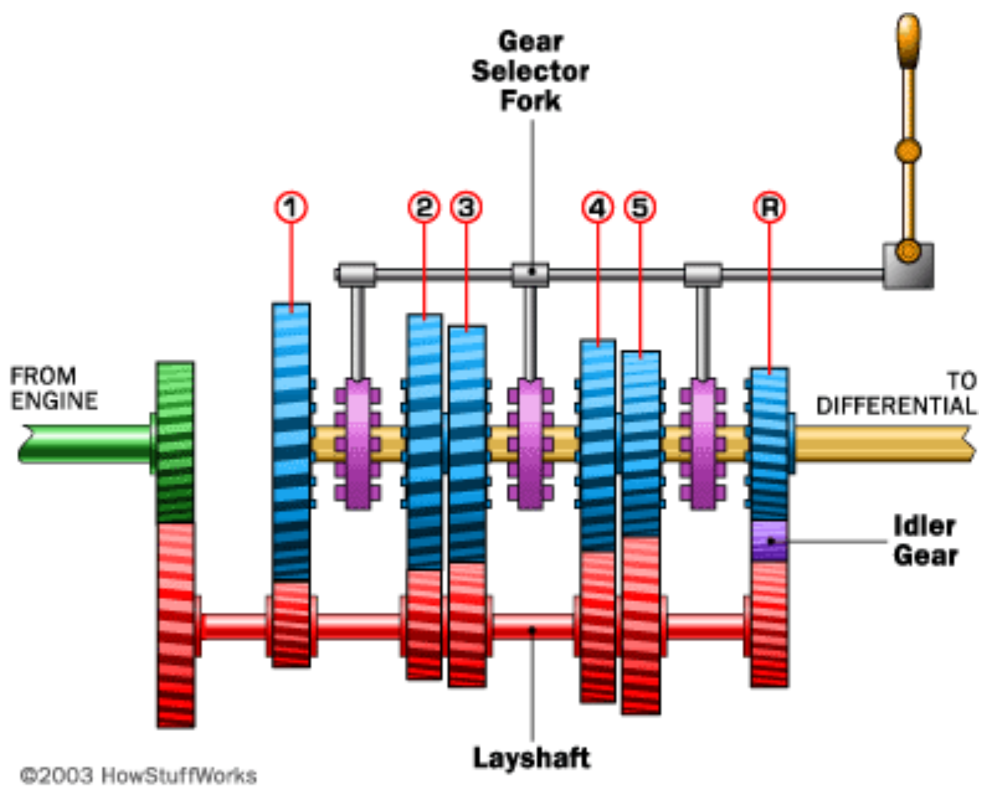
Joint work with Pavithra Prabhakar

Hybrid system

A dynamical system exhibiting a mixed discrete and continuous behavior.



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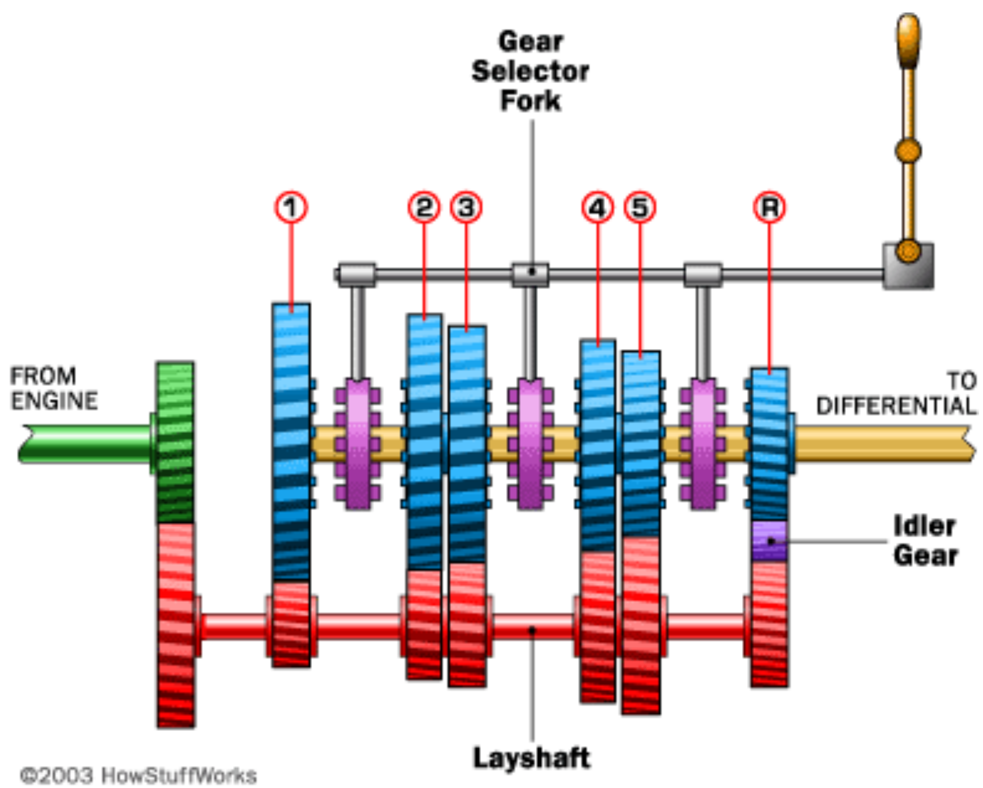
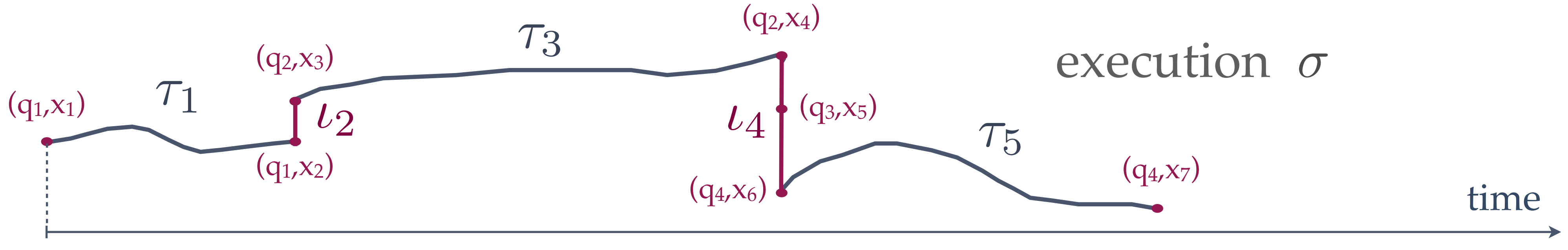


Hybrid system

$\mathcal{H} = (Q, X, \Sigma, \Delta)$

- Q finite set of control modes (discrete state space),
- $X = \mathbb{R}^n$ continuous state space,
- $\Sigma \subseteq \mathit{Trans}(Q, X)$ set of transitions and
- $\Delta \subseteq \mathit{Traj}(Q, X)$ set of trajectories.



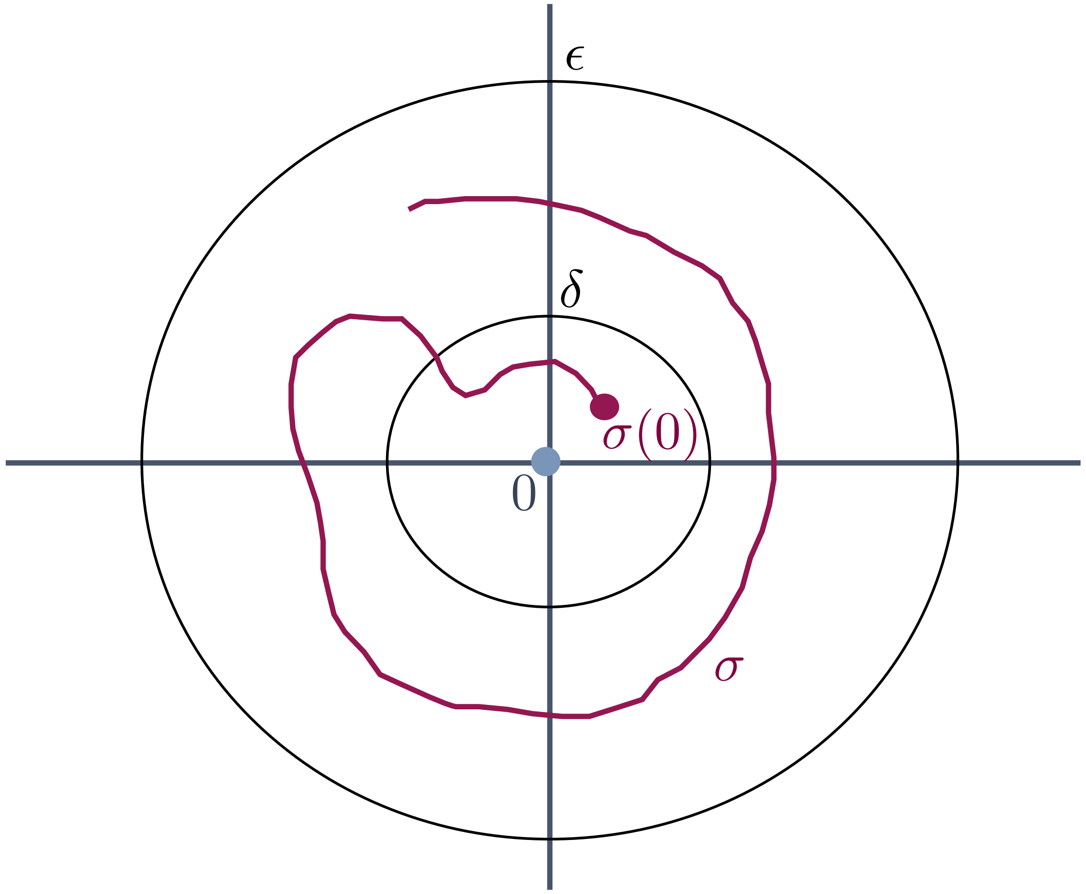
Stability

- Stability is a fundamental property in control system design and captures robustness of the system with respect to initial states or inputs.

- A system is stable when small perturbations in the input just result in small perturbations of the eventual behaviours.

- Classical notions of stability:
  - Lyapunov stability
  - Asymptotic stability

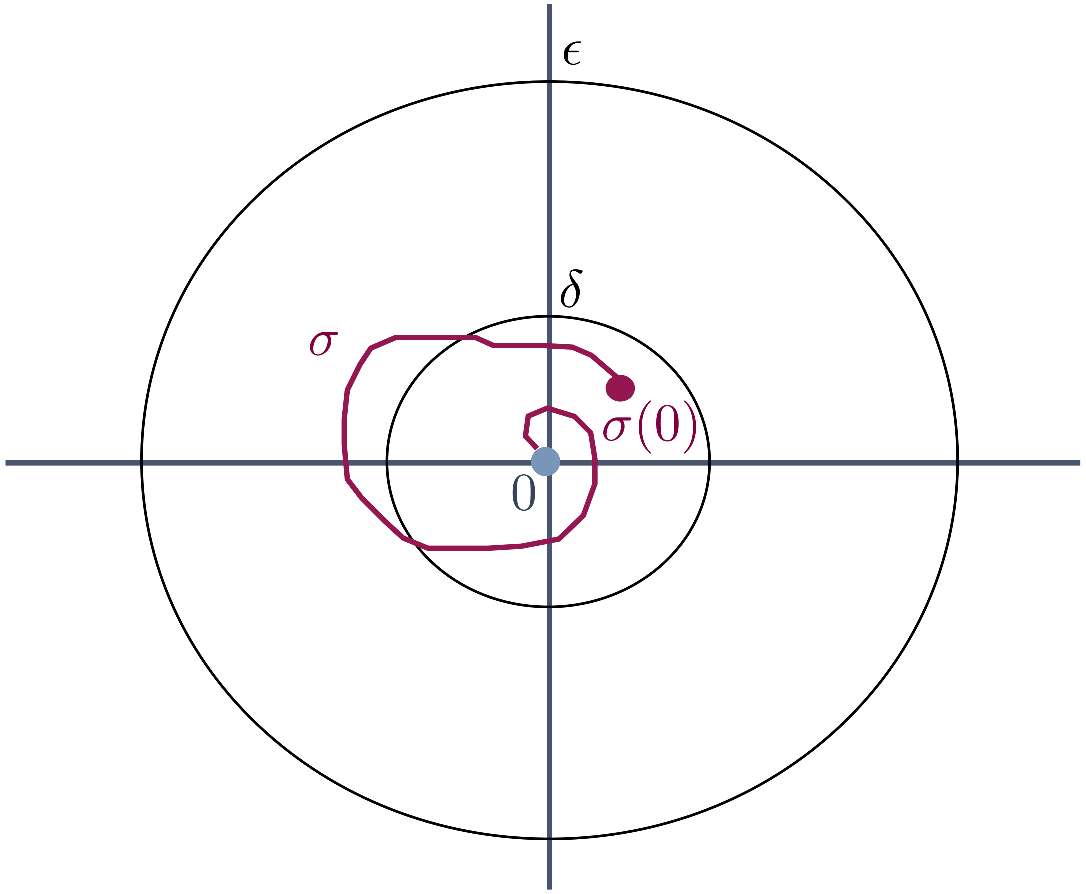
Lyapunov stability



The equilibrium point 0 is Lyapunov stable if

$$\forall \epsilon > 0 \ \exists \delta = \delta(\epsilon) > 0 : ||\sigma(0)|| < \delta \Rightarrow ||\sigma(t)|| < \epsilon \ \ \forall t \geqslant 0$$

Asymptotic stability



The equilibrium point 0 is asymptotic stable if it is Lyapunov stable and every execution converges to 0.

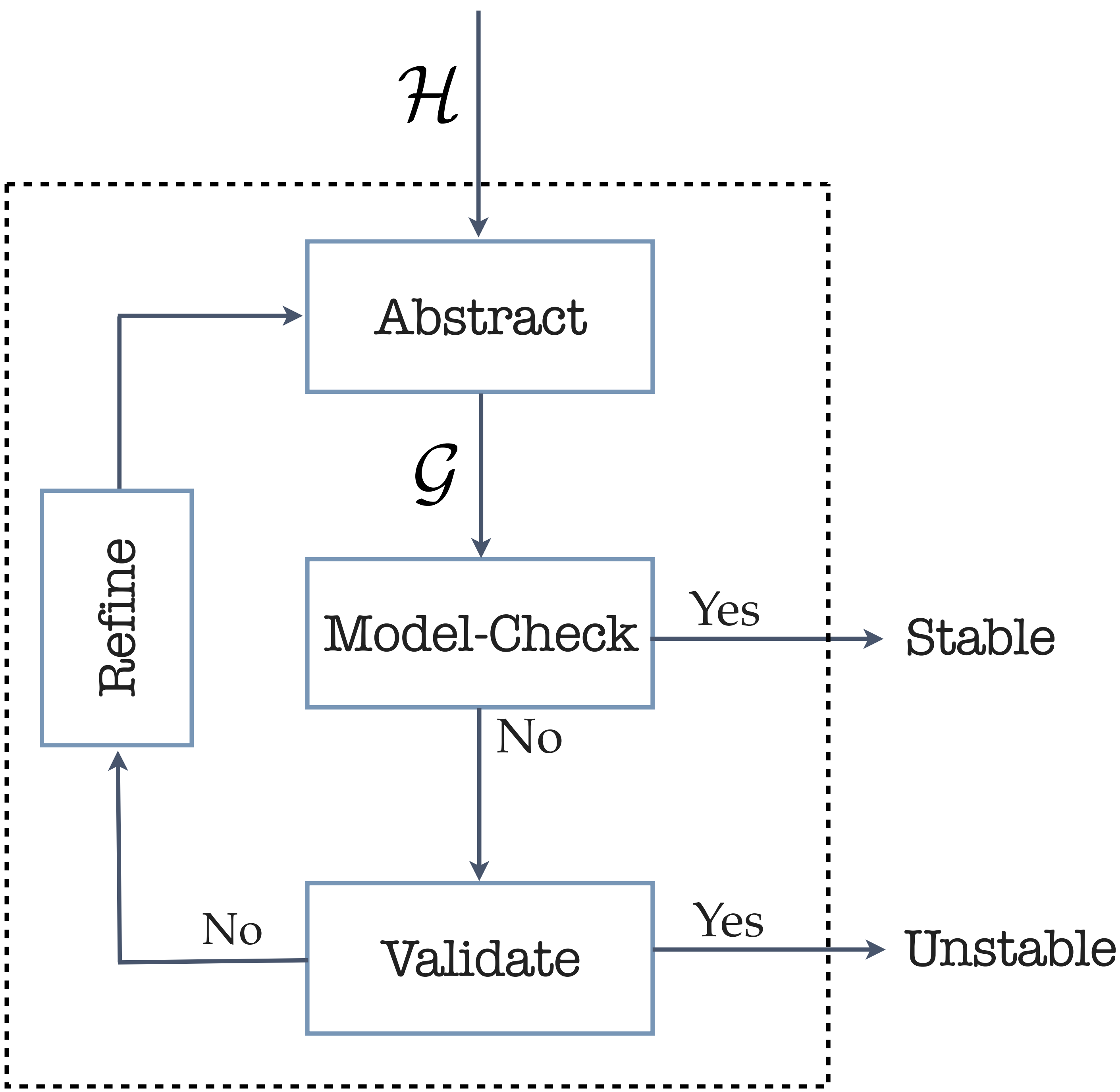
State of the art

- Existence of Lyapunov function assures stability.
- Lyapunov function computation:
  - Choose a template: $L(x) = ax^2 + bx + c$.
  - Look for coefficients a, b, c, such that $L(x)$ holds some conditions.
  - If a, b, c do not exist, choose a new template.
- Template choice requires user ingenuity.
- Coefficient failure does not provide insights on the next template choice.

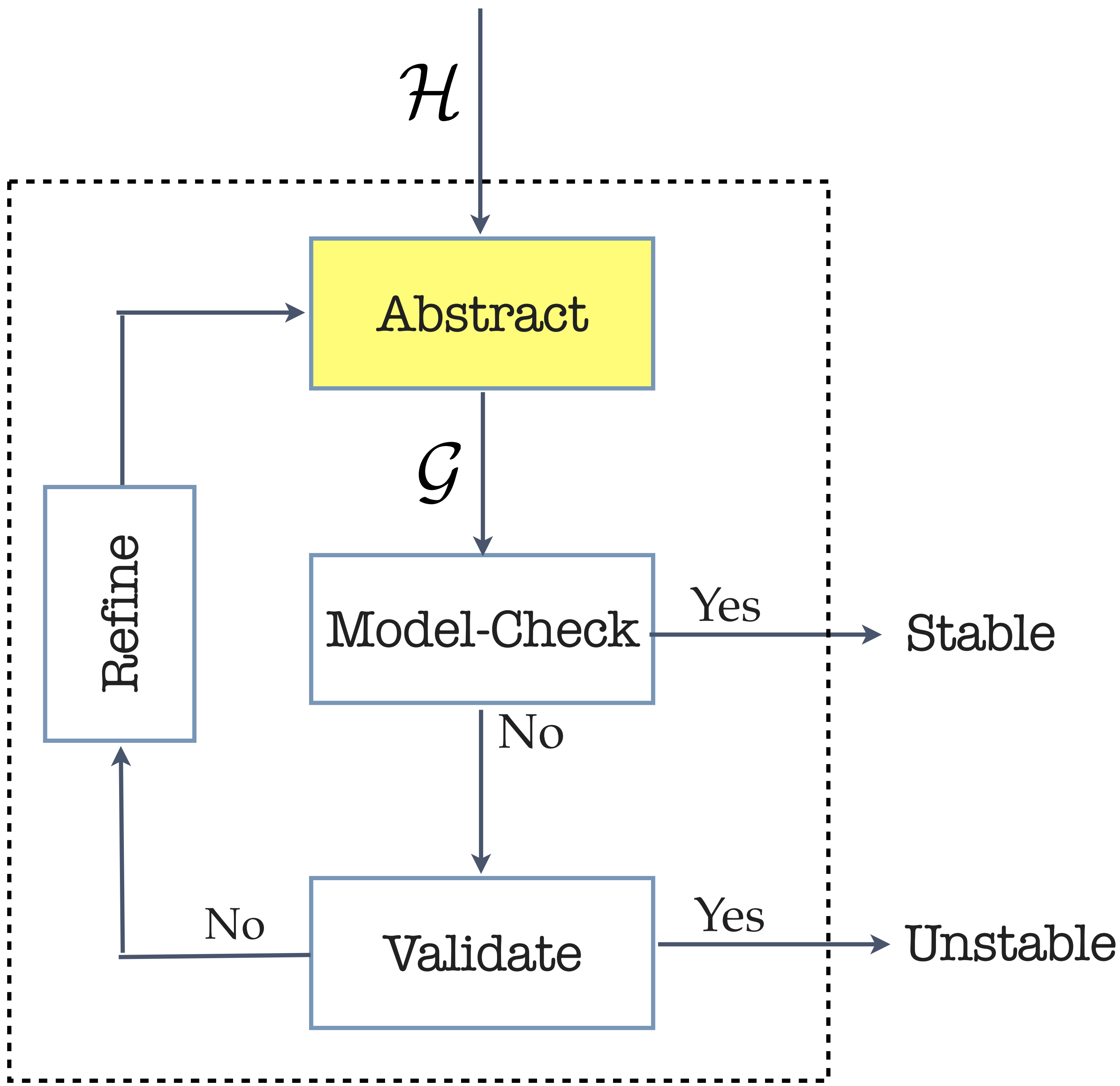
Motivation

- Automatization of stability analysis.
- Development of an abstraction refinement framework.

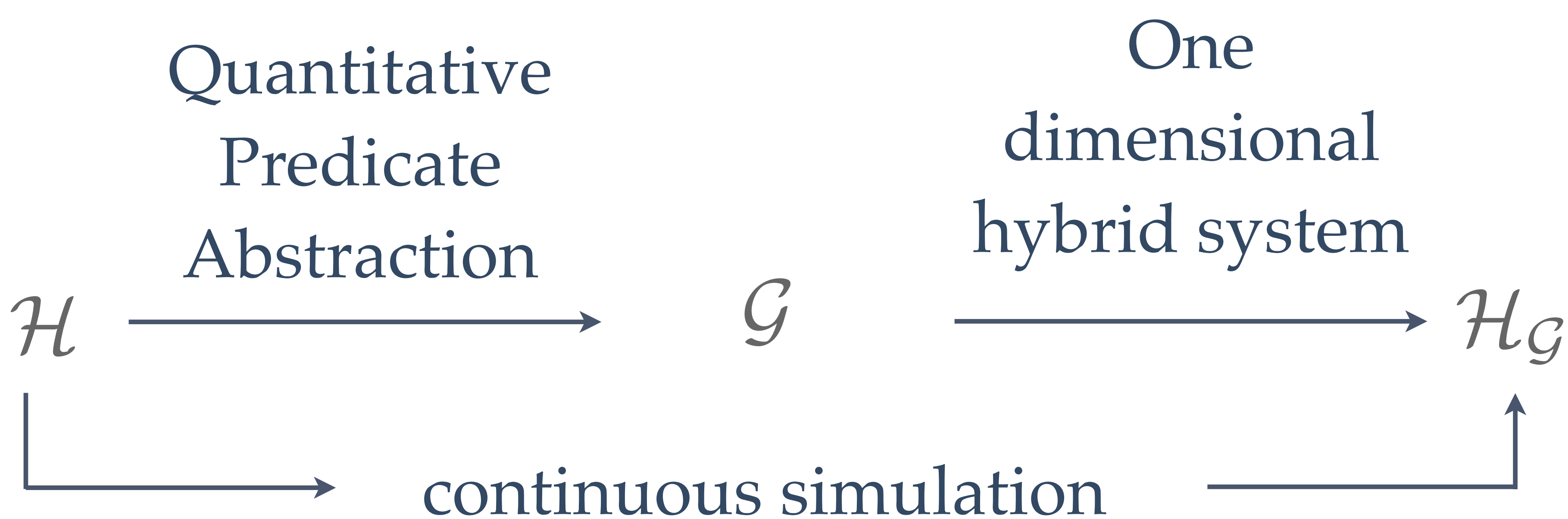
Algorithmic approach



Abstraction



Theoretical foundation



Continuous simulation

Let R be a continuous simulation from a hybrid system $\mathcal{H}$ to a hybrid system $\mathcal{H}_\mathcal{G}$. Then:

- $\mathcal{H}_\mathcal{G}$ Lyapunov stable $\Rightarrow$ $\mathcal{H}$ Lyapunov stable
- $\mathcal{H}_\mathcal{G}$ asymptotically stable $\Rightarrow$ $\mathcal{H}$ asymptotically stable

Quantitative predicate abstraction

- Abstraction based on predicates.
- In addition, weight computation.

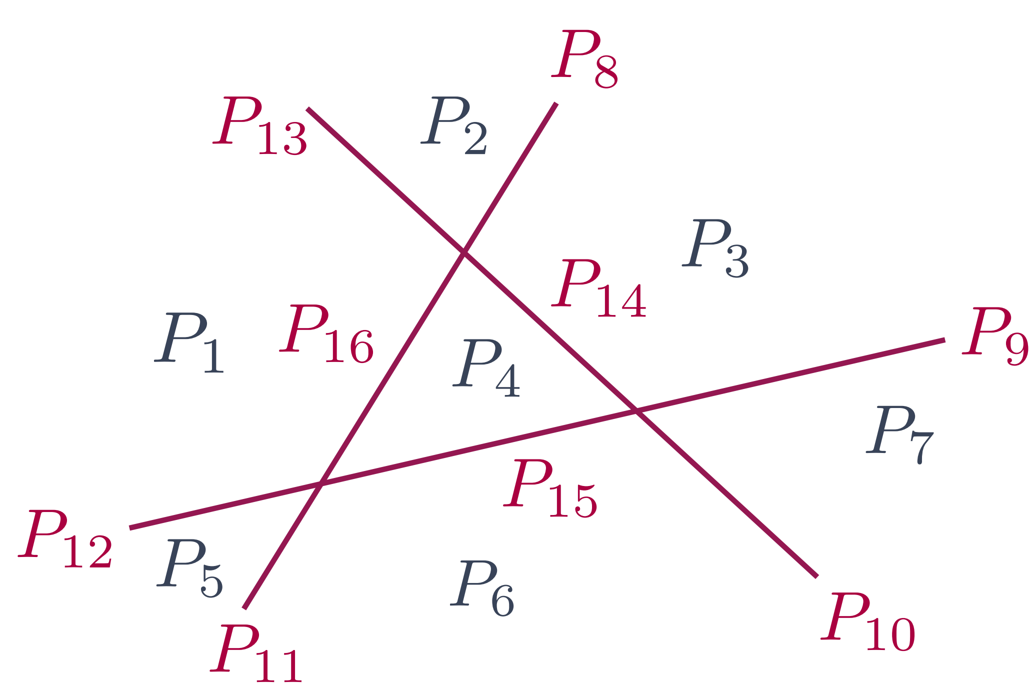
Partition

$\mathcal{H} = (Q, X, \Sigma, \Delta)$ Hybrid system

$\mathcal{P} = \{P_1, \cdots, P_k\}$ Polyhedral partition of X such that:

- $X = \bigcup_{i=1}^{k} P_i$
- $Int(P_i) \cap Int(P_j) = \emptyset \;\; \forall i \neq j$

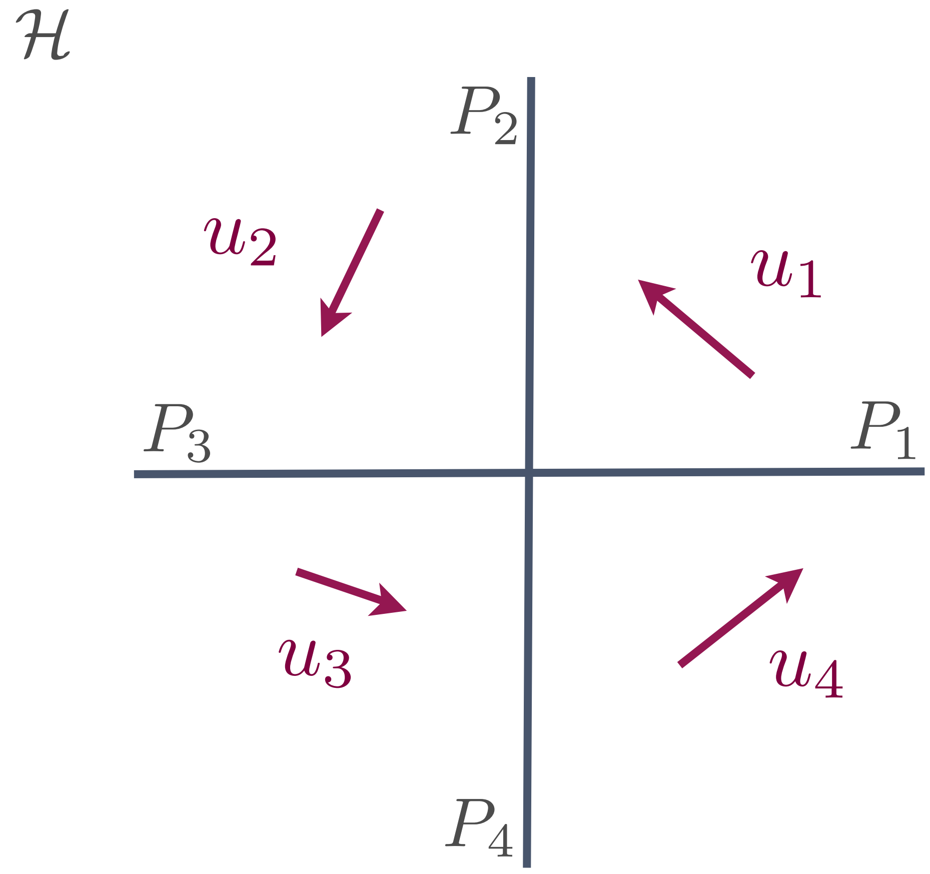
Regions $= \mathcal{P}$



Quantitative predicate abstraction

- Modified predicate abstraction resulting in a finite weighted graph, $\mathcal{G}$.
- Nodes correspond to the regions of the partition, $\mathcal{P}$.
- Edges represent existence of an execution from one region to other and evolving through a common adjacent region.
- Weight on every edge corresponds to the maximum scaling of possible executions.

Predicate abstraction: constant derivative



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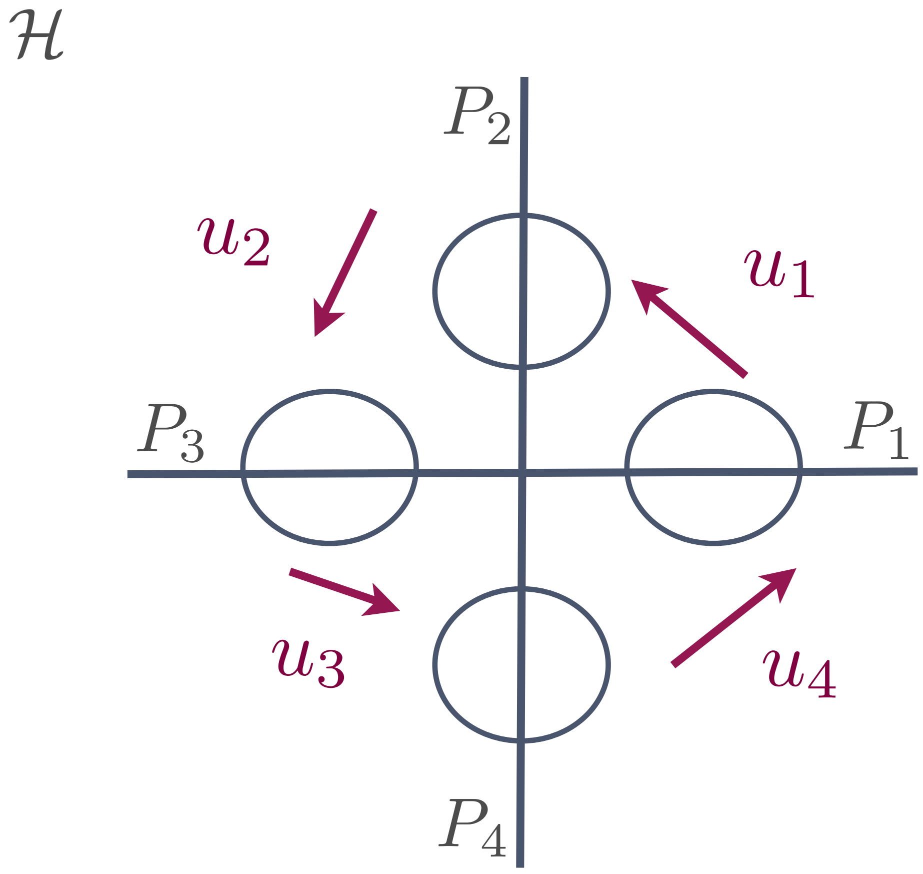

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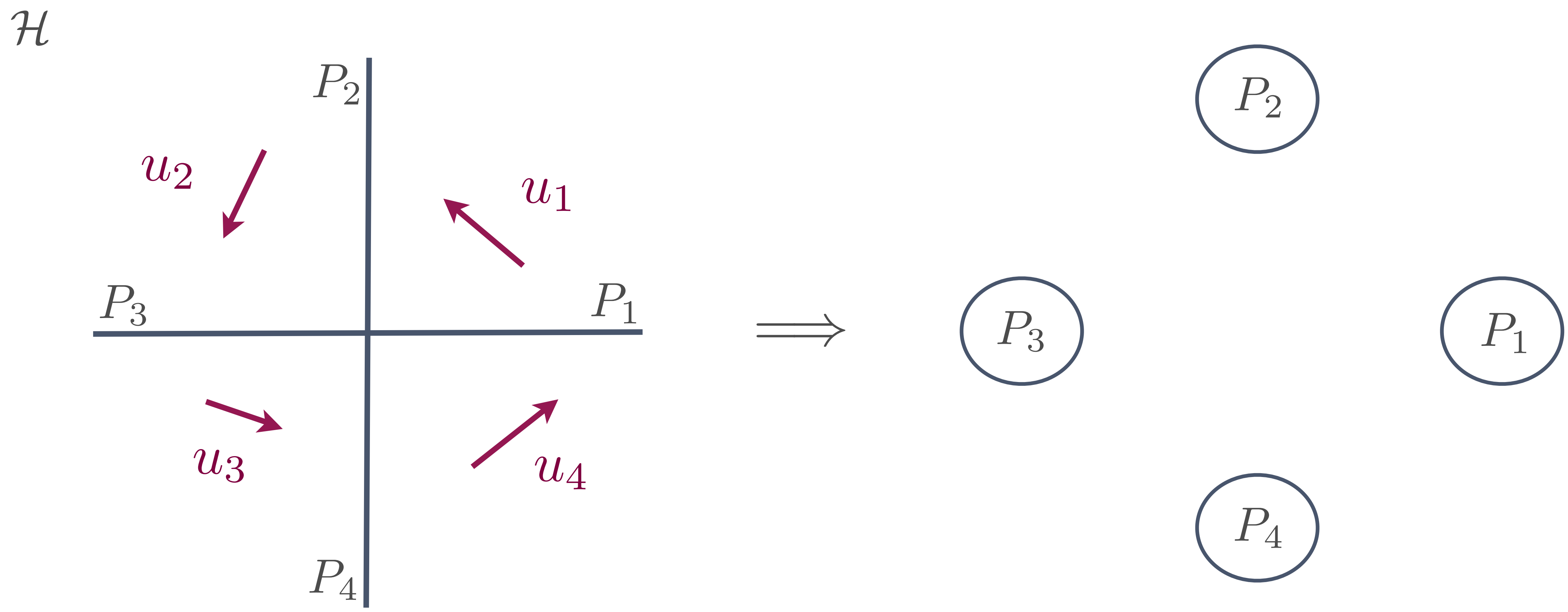

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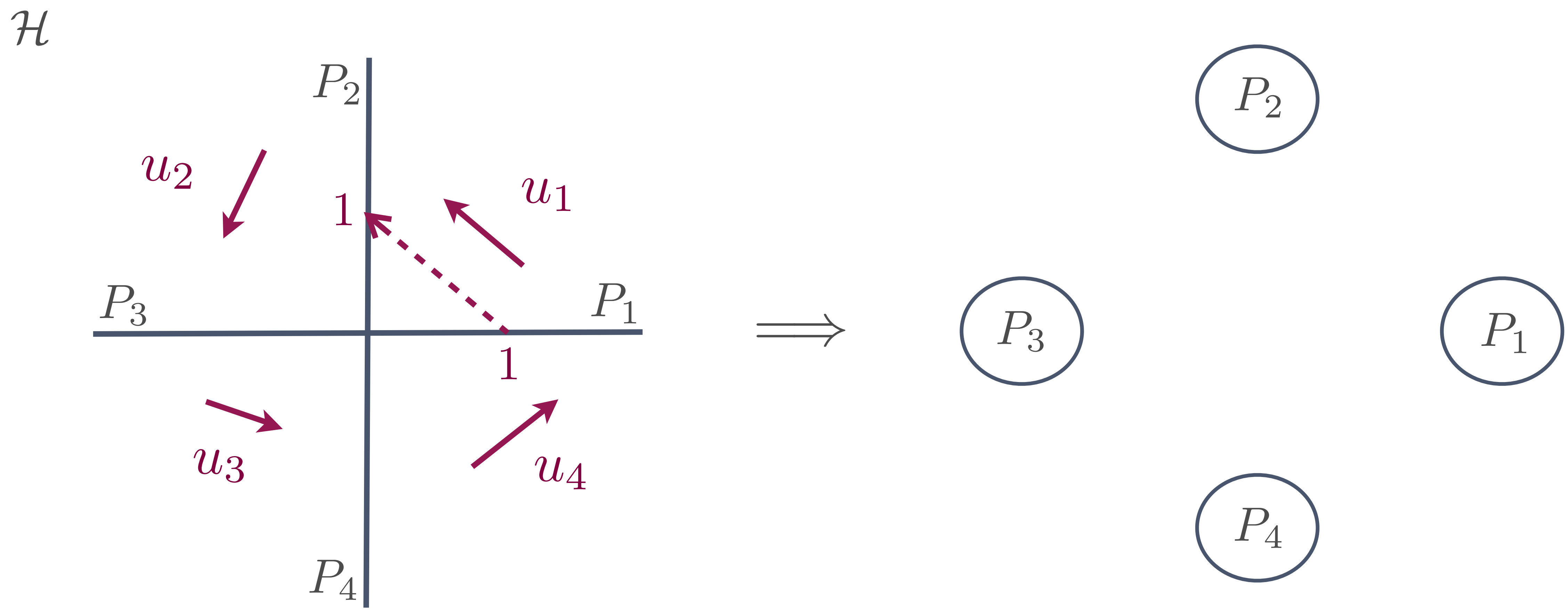
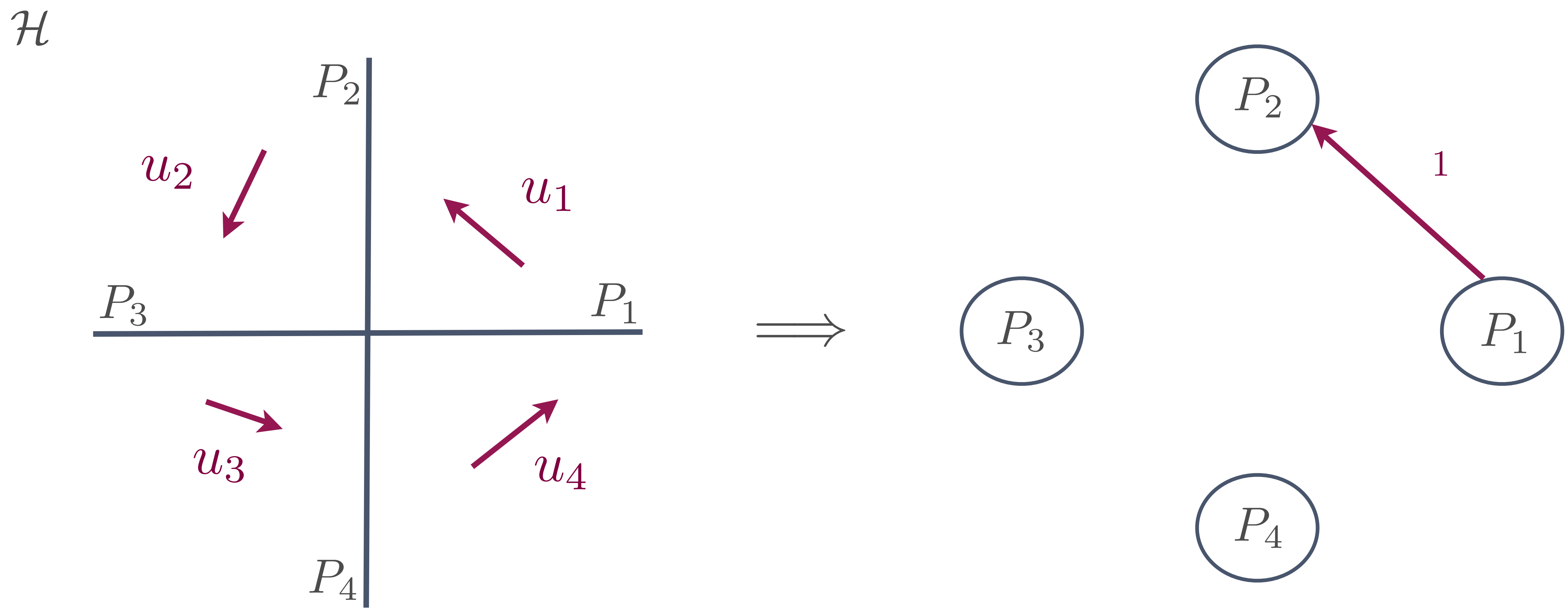


Predicate abstraction: constant derivative



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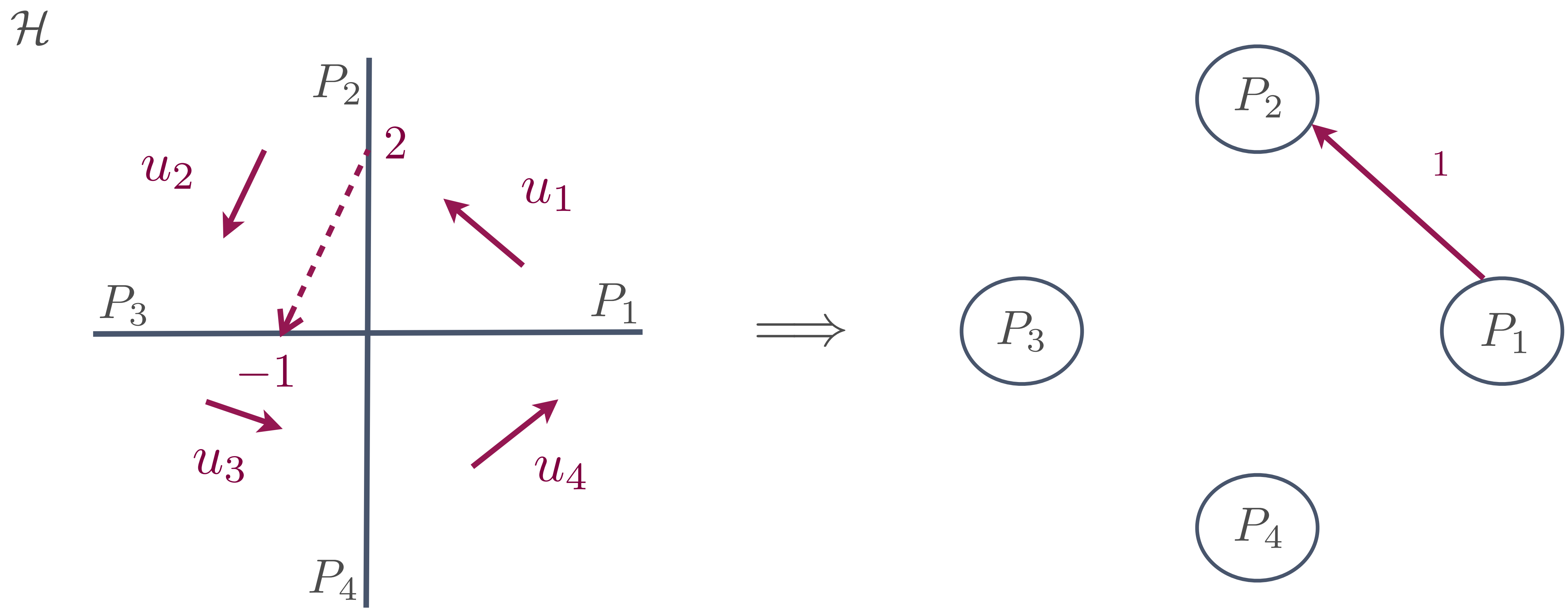
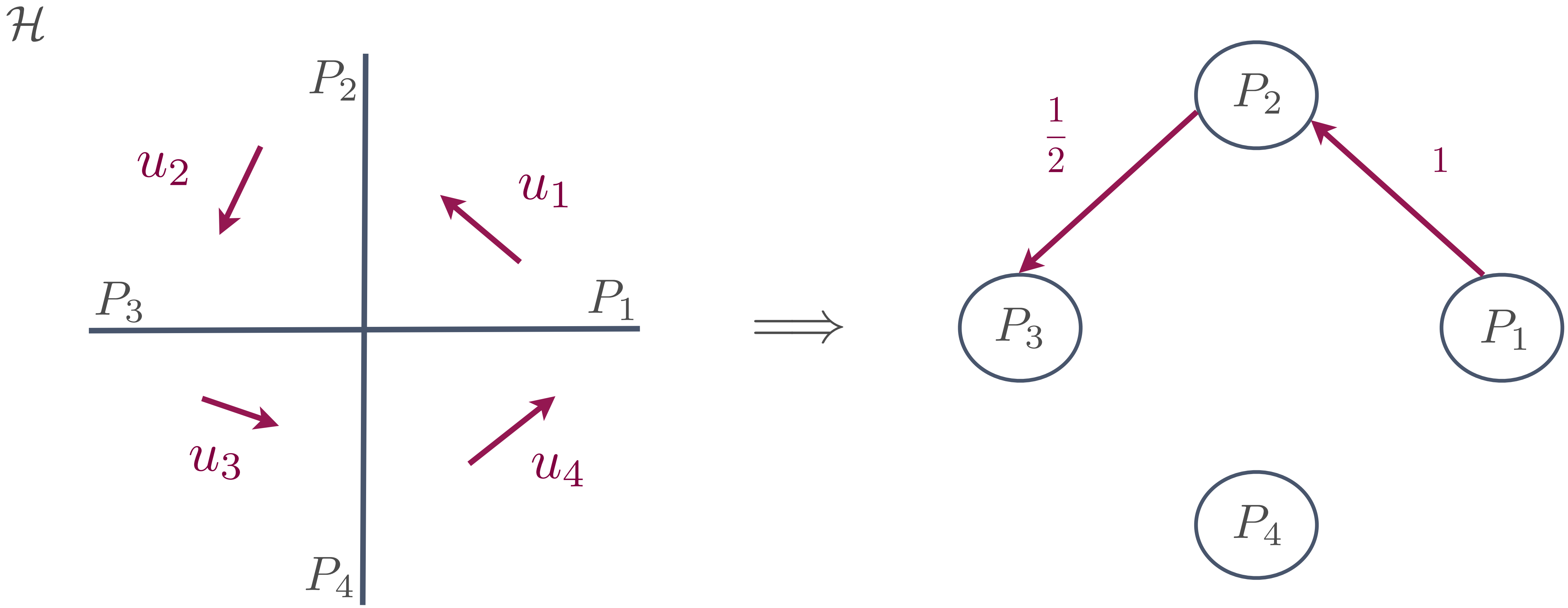


Predicate abstraction: constant derivative



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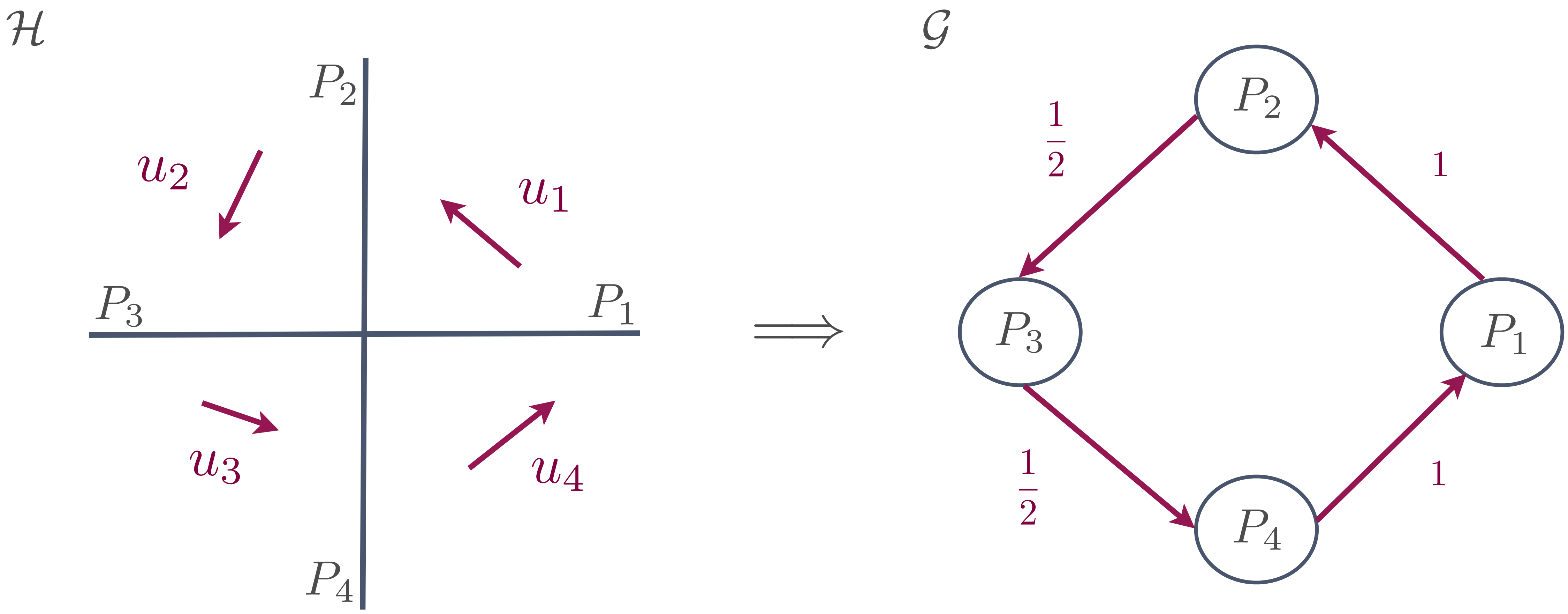
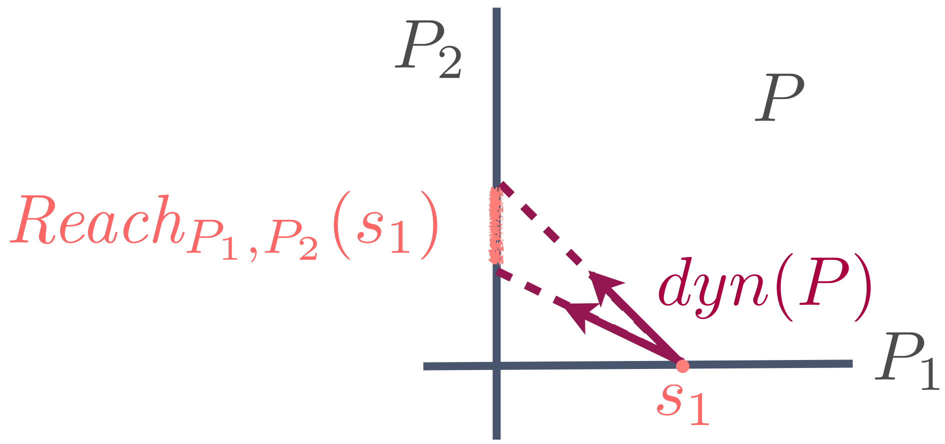


Reachability relation

$(s_1, s_2) \in \mathit{ReachRel}_{P_1,P_2}$ if there exists an execution σ:

- $\sigma(0) = s_1 \in P_1$,
- $\exists\, T \geqslant 0$ with $\sigma(T) = s_2 \in P_2$ and
- $\exists\, P \in \mathcal{P}$ such that $\forall t \in (0, T)$, $\sigma(t) \in P$.

Reachability relation - polyhedral dynamics



- Polyhedral hybrid system:

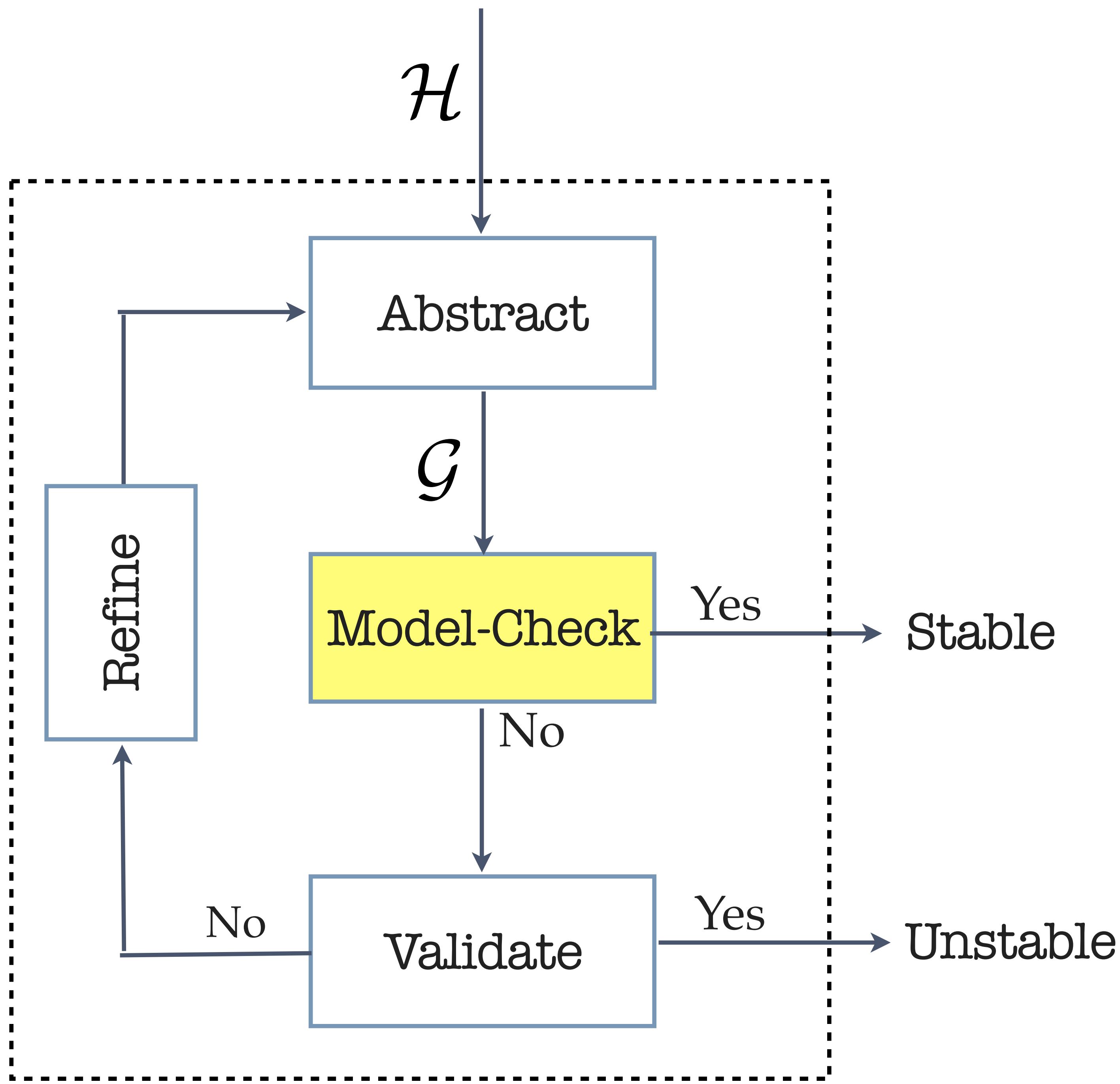
$$ReachRel_{P_1,P_2} = \{(s_1, s_2) : s_1 \in P_1, s_2 \in P_2, \exists t, \exists u \in dyn(P)$$

$$\text{for some } P \text{ such that } s_2 = s_1 + ut\}$$

Weight computation

$$W(P_1, P_2) = \sup_{(s_1, s_2) \in ReachRel_{P_1, P_2}} \frac{||s_2||}{||s_1||}$$

Model-checking



Model-checking

Let $\mathcal{G}$ be a quantitative abstraction of a hybrid system $\mathcal{H}$.

G1 There is no edge e in $\mathcal{G}$ with infinite weight.

G2 The product of the weights on every simple cycle π of $\mathcal{G}$ is less than or equal to 1.

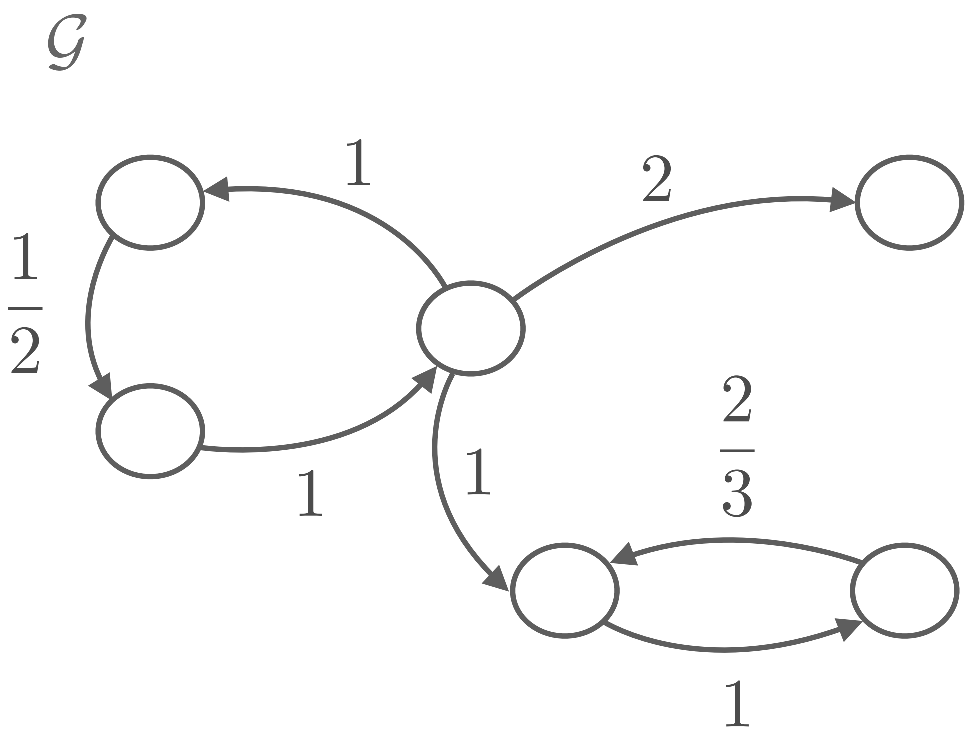
G3 Every node in $\mathcal{G}$ is labelled by "conv".

G4 The product of the weights on every simple cycle π of $\mathcal{G}$ is strictly less than 1.

Then:

- $\mathcal{H}$ is Lyapunov stable if conditions G1 and G2 hold; and
- $\mathcal{H}$ is asymptotically stable if conditions G3 and G4 hold.

Model-checking



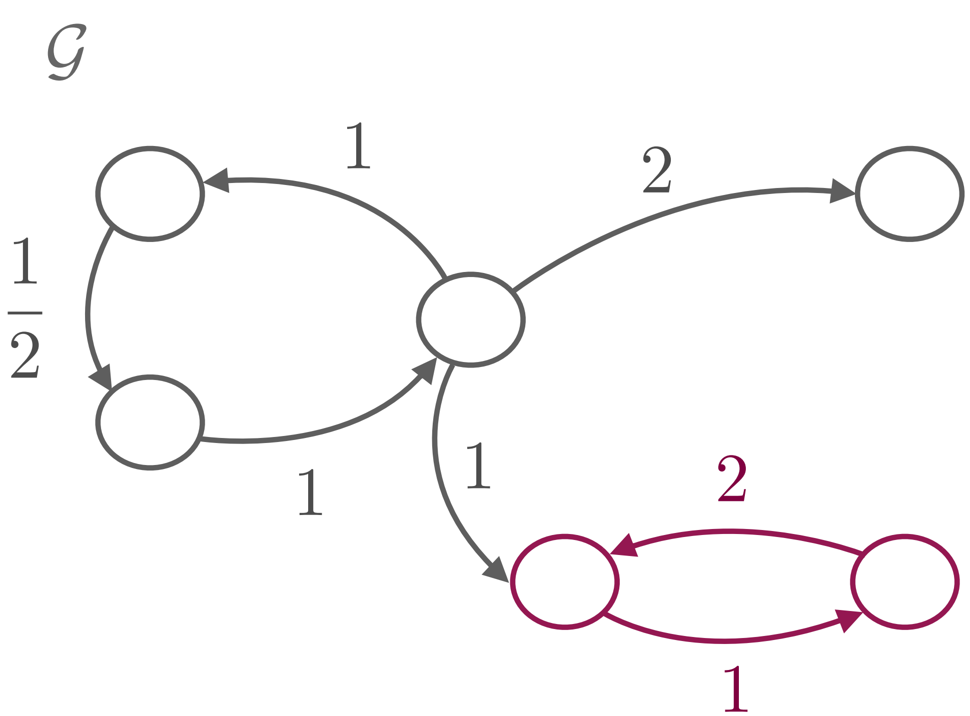
Every cycle has weight smaller than 1

$\Downarrow$

$\mathcal{H}$ is stable

$\Downarrow$

STOP



There is a cycle, π, with weight greater than 1

$\Downarrow$

π is a counterexample

AVERIST

Software tool

- Quantitative predicate abstraction for polyhedral switched systems.
- Stability analysis based on the weighted graph.

- Implemented in Python.
- Parma Polyhedra Library (PPL) to manipulate polyhedral sets.
- GLPK solver to compute the weights.
- NetworkX Python package to define and analyse graphs.

`http://software.imdea.org/projects/averist/index.html`

Conclusions

- Summary of an algorithmic approach for stability verification.
- Future directions:
  - Extension to linear and nonlinear dynamics.
  - Compositional techniques for stability analysis.

Thank you!