A CEGAR Approach for Stability Verification of Linear Hybrid Systems

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Co-authored work with Pavithra Prabhakar

DARS 2017

Cyber-Physical Systems (CPSs)

Systems in which software "cyber" interacts with the "physical" world

Medical Devices

Automotive

Robotics

Aeronautics

Process control

Software controlled physical systems

- Automotive systems: Cruise control, lane assistants
- Medical Devices: Pacemakers, infusion pumps

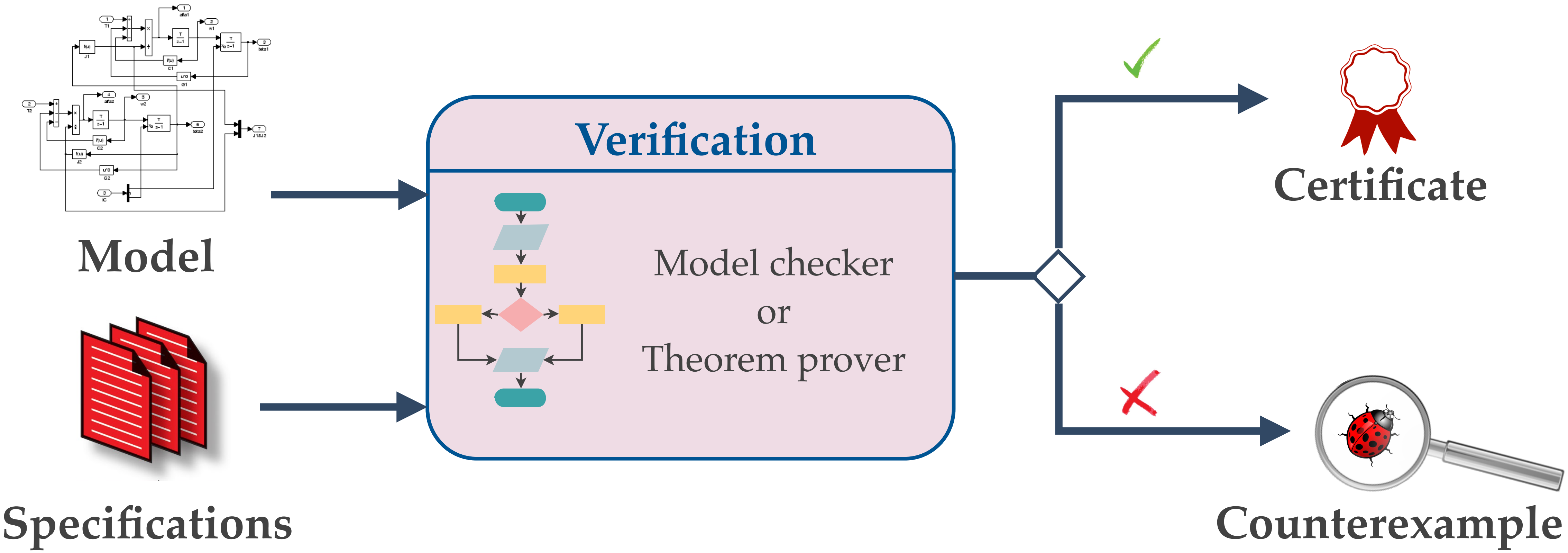
Critical aspects in CPS design

- Security
- Reliability
- Safety

Grand Challenge

How do we build and deploy robust CPS?

Formal Verification

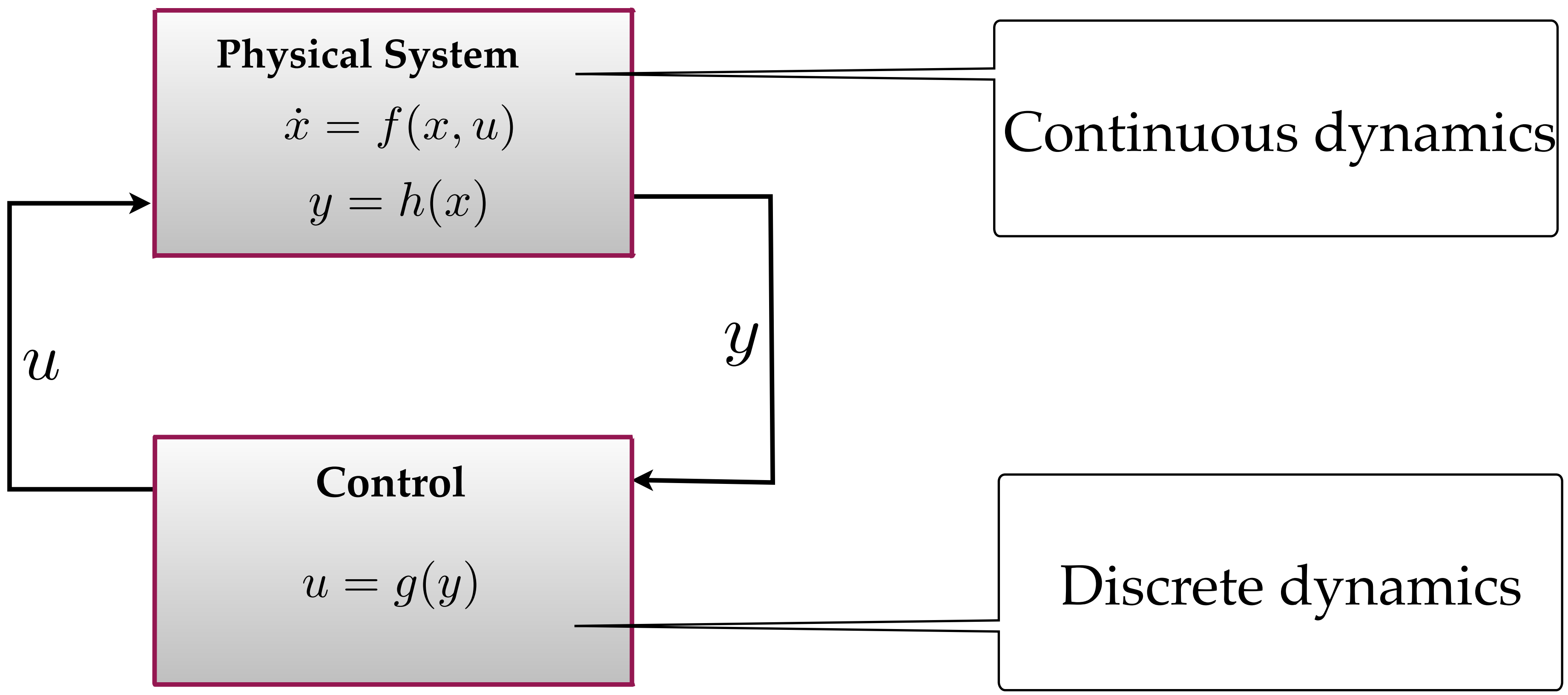


- Models for Cyber-Physical Systems (Automata based)
- Robustness Specifications (Logic based)
- Verification Algorithms (Model checker)

CPS Model

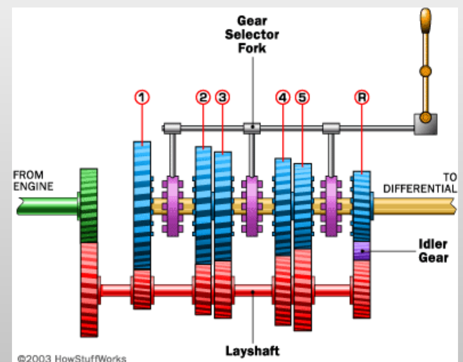
Hybrid Control Systems

Hybrid Systems capture one of the main features of CPS, the mixed **continuous** and **discrete** behaviour.



Cruise control & automatic gearbox

Gearbox



$$\dot{v} = \frac{p_q^r T}{M}$$

v

T, q

Cruise control

PI Control

$$\frac{K_q}{\tau}\int (v_d - v)dv$$

$$K_q(v_d - v)$$

q

Discrete Control

if $v - v_d = \frac{1}{p_i}\omega_{low}$
$q \to q - 1$

if $v - v_d = \frac{1}{p_i}\omega_{high}$
$q \to q + 1$

v_d

Discrete Variable

Gear Position q

$$q = 1, 2, 3, 4$$

Continuous Variables

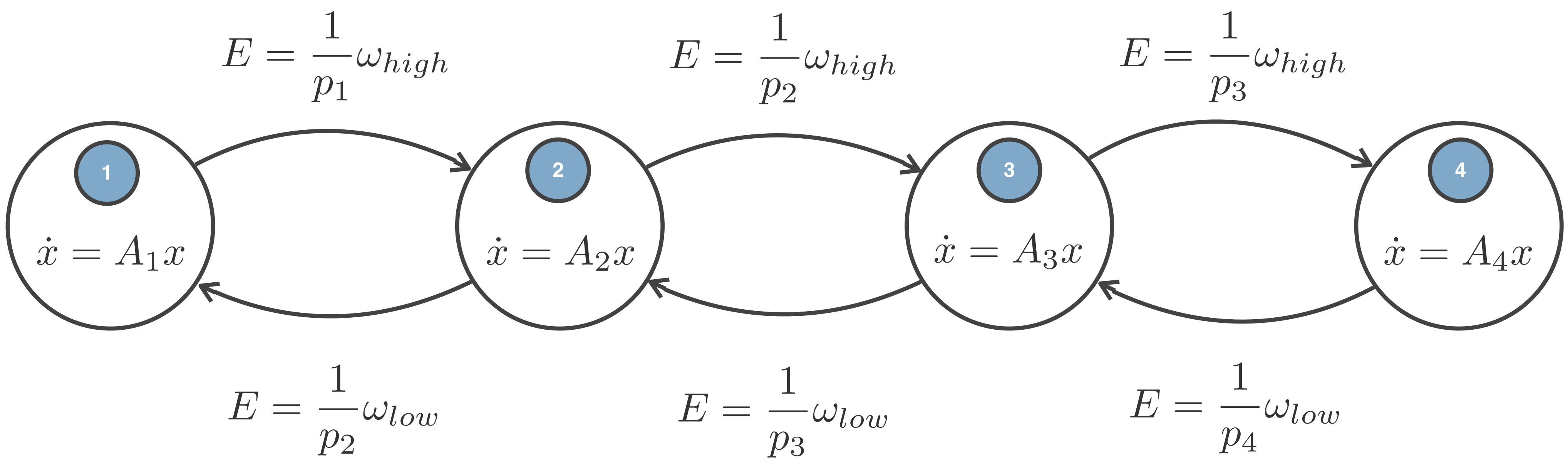
Error $E = (v_d - v)$

Torque T

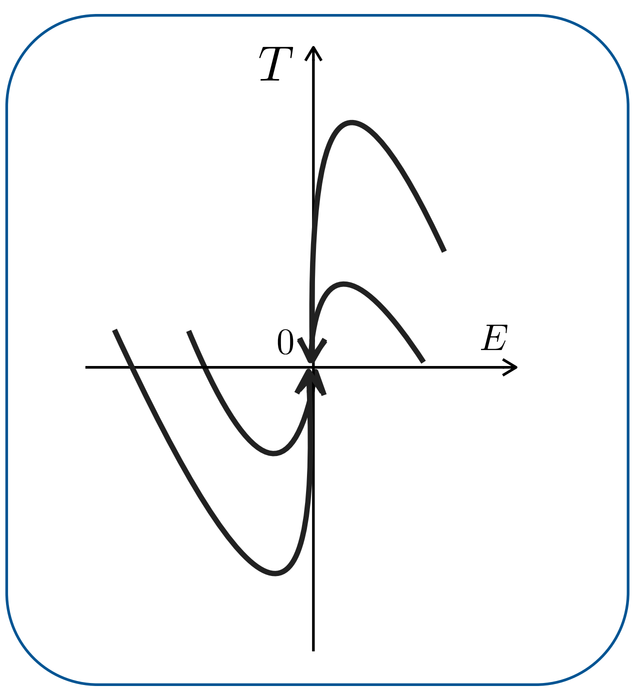
Continuous Dynamics

$$\dot{E} = \frac{-p_q^r}{M}T$$

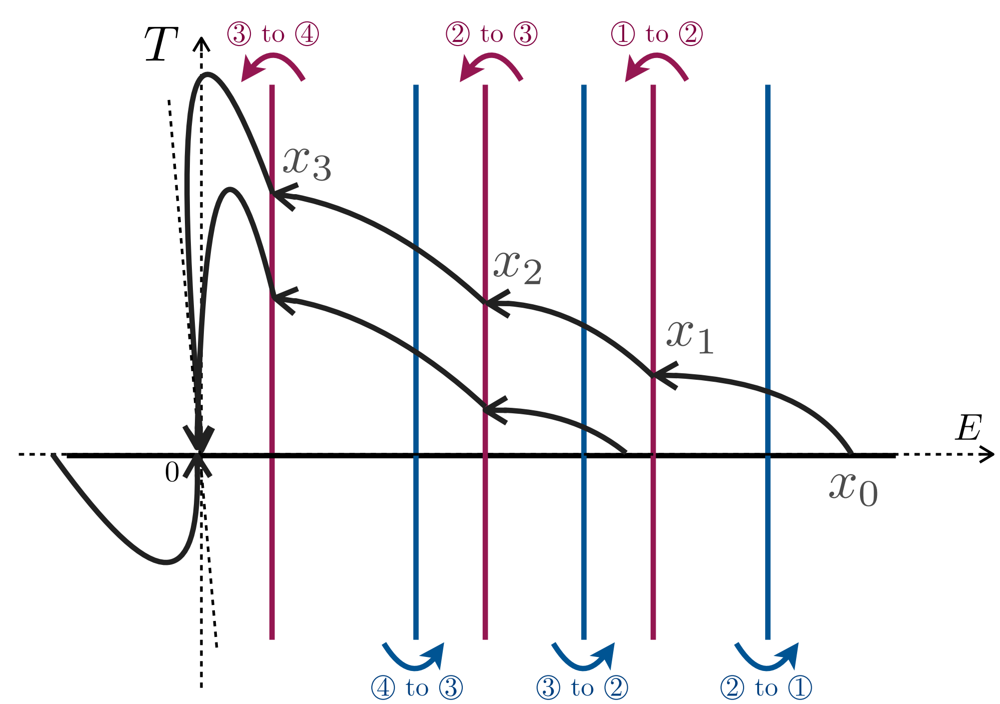
$$\dot{T} = \frac{K_q}{r}E + K_q E$$



Trajectories



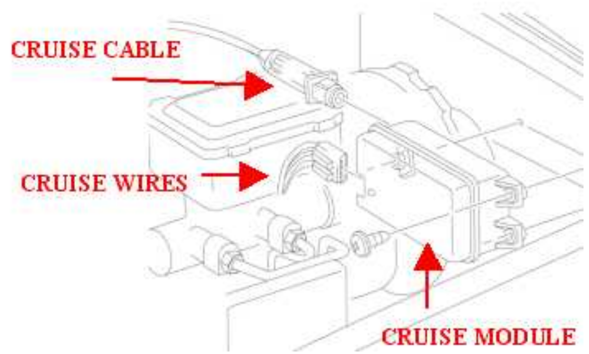
Executions



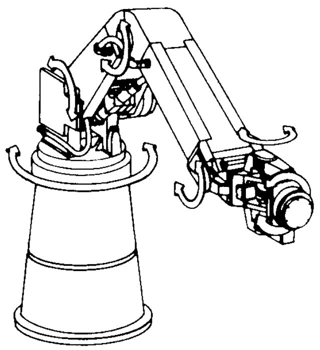
CPS Specifications

Specifications

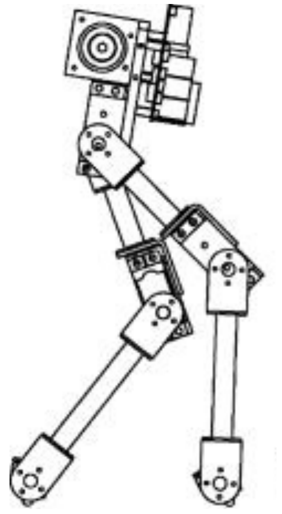
Stability: Small perturbations in the initial state or input to the system result in only small deviations from the nominal behavior



Cruise control



Robotic arm

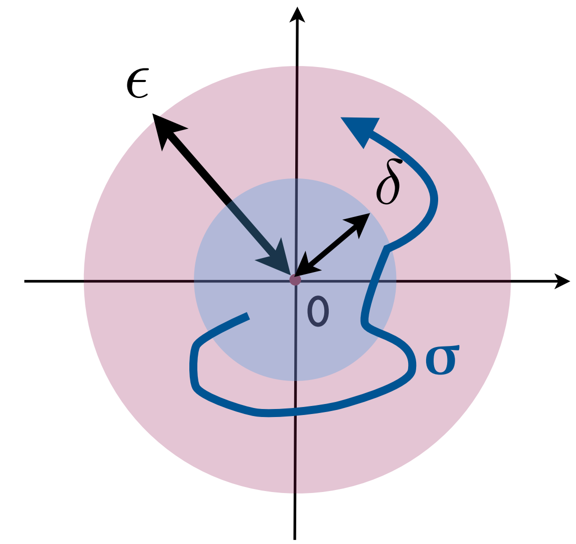


Bipedal robot walking

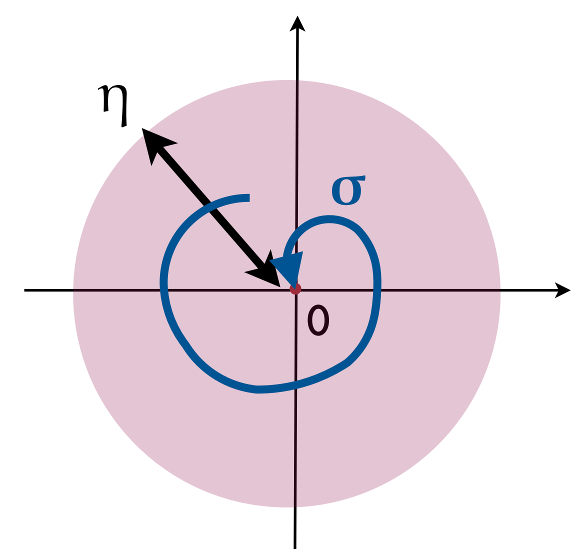
- Cruise control: stability with respect to the desired velocity
- Robotic arm: stability with respect to the set point
- Bipedal walking: stability with respect the periodic orbit

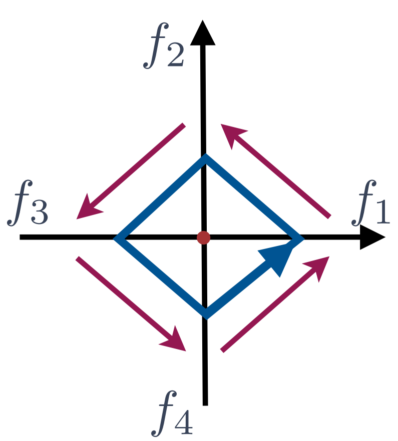
Stability notions

A system is **Lyapunov stable** with respect to the **equilibrium point 0** if for every $\varepsilon > 0$ there exists $\delta > 0$ such that for every execution σ starting from $B_\delta(0)$, $\sigma(t) \in B_\varepsilon(0)$, for all time t.

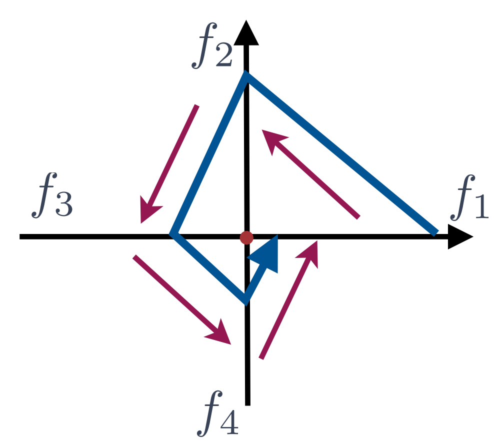


A system is **asymptotically stable** with respect to the **equilibrium point 0** if it is Lyapunov stable and there exist $\eta > 0$ such that every execution σ starting from $B_\eta(0)$ converges to 0.

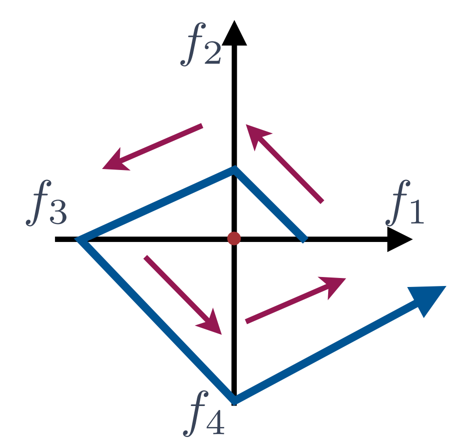




Lyapunov Stable



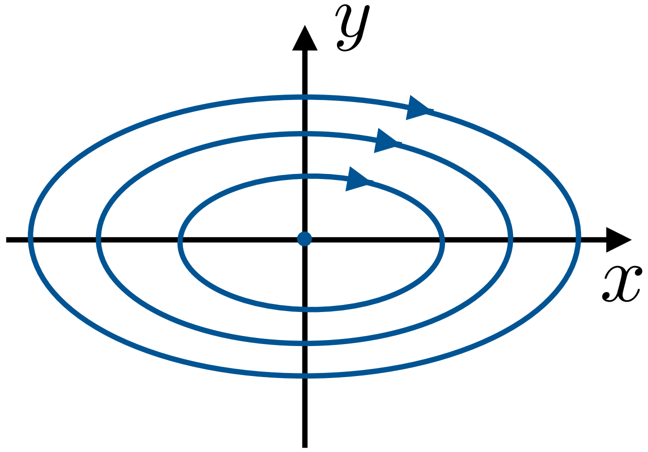
Asymptotically Stable



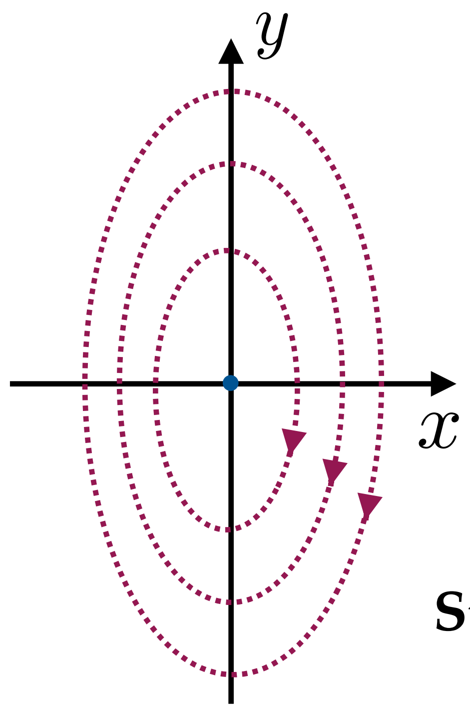
Unstable

Stability analysis challenges

Linear dynamical systems



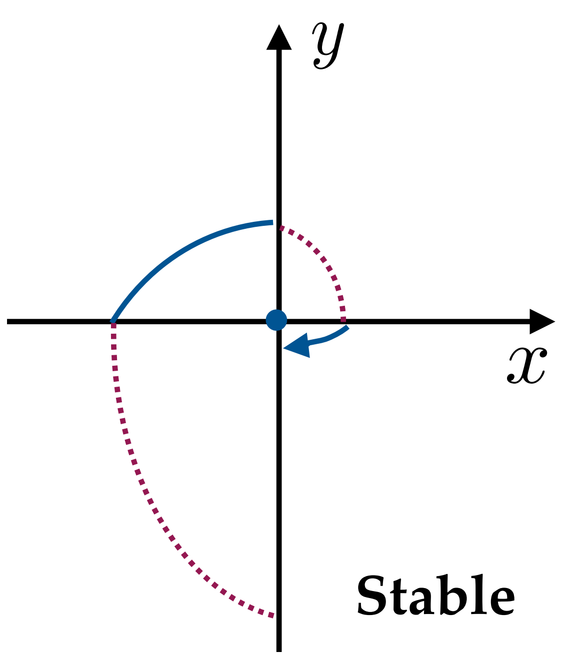
Stable



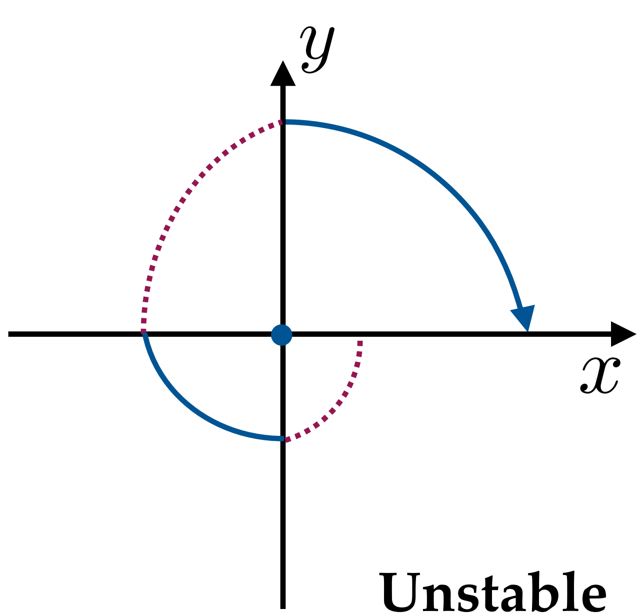
Stable

Stability can be determined by eigenvalues analysis

Linear hybrid systems



Stable



Unstable

Eigenvalue analysis does not suffice for switched linear system

State of the art: Lyapunov's second method

Continuous dynamics:

$$\dot{x} = F(x)$$

If there exists a **Lyapunov function** for the system, then the system is Lyapunov stable

<u>Lyapunov function</u>

- Continuously differentiable

$$V : \mathbb{R}^n \to \mathbb{R}^+$$

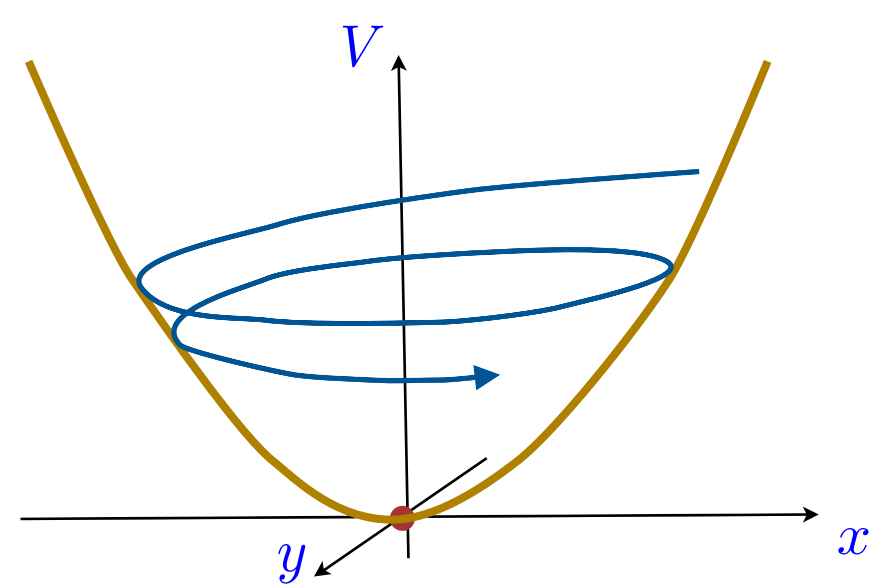
- Positive definite

$$V(x) \geq 0 \, \forall x$$

$$V(x) = 0 \text{ iff } x = 0$$

- Function value decreases along any trajectory

$$\frac{\partial V(x)}{\partial x} F(x) \leq 0 \; \forall x$$



Switched and hybrid systems:

- Common Lyapunov functions
- Multiple Lyapunov functions

Automated analysis

Template based automated search

- Choose a template
- Encode Lyapunov function conditions as constraints
- Solve using sum-of-squares programming tools

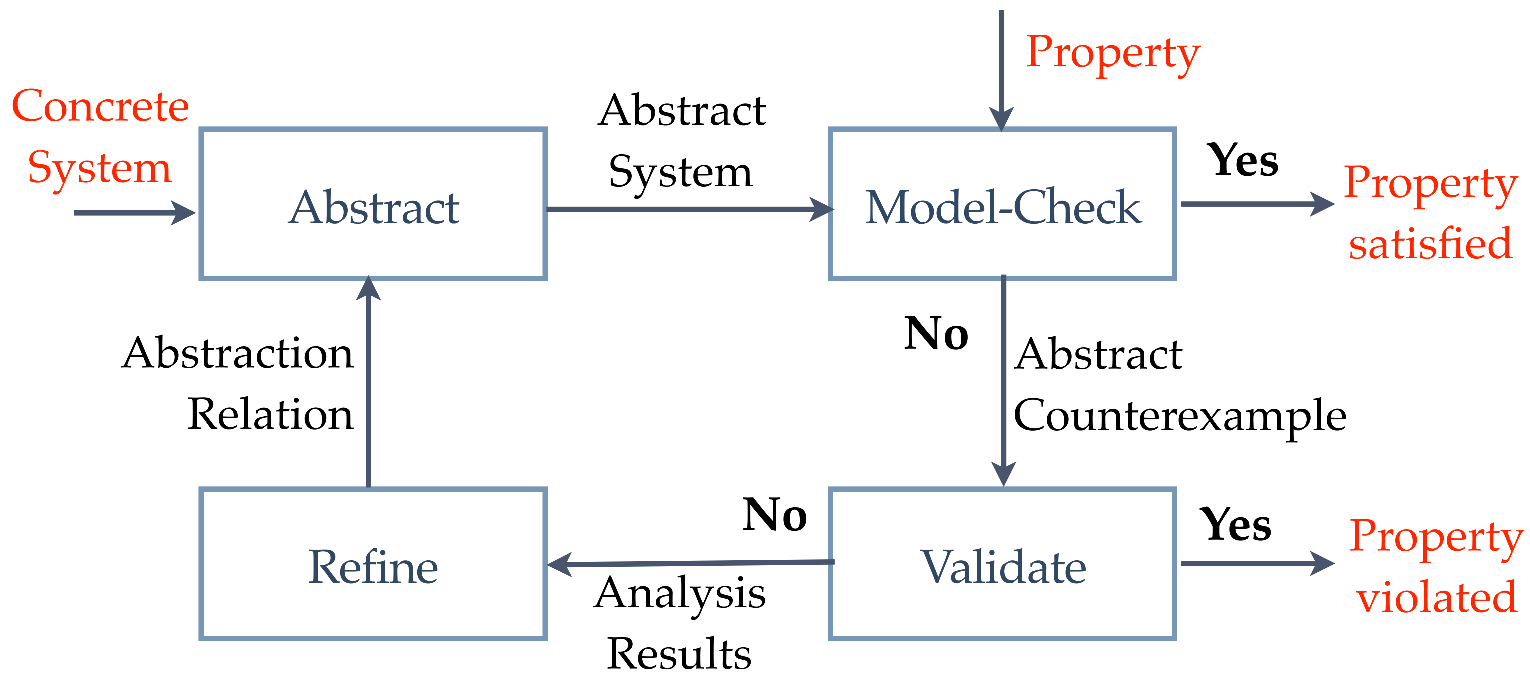
Shortcomings:

- Success depends crucially on the choice of the template
- The current methods provide no insight into the reason for the failure, when a template fails to prove stability
- No guidance regarding the choice of the next template

Alternate approach
CEGAR

Counterexample Guided Abstraction Refinement (CEGAR)

CEGAR for stability

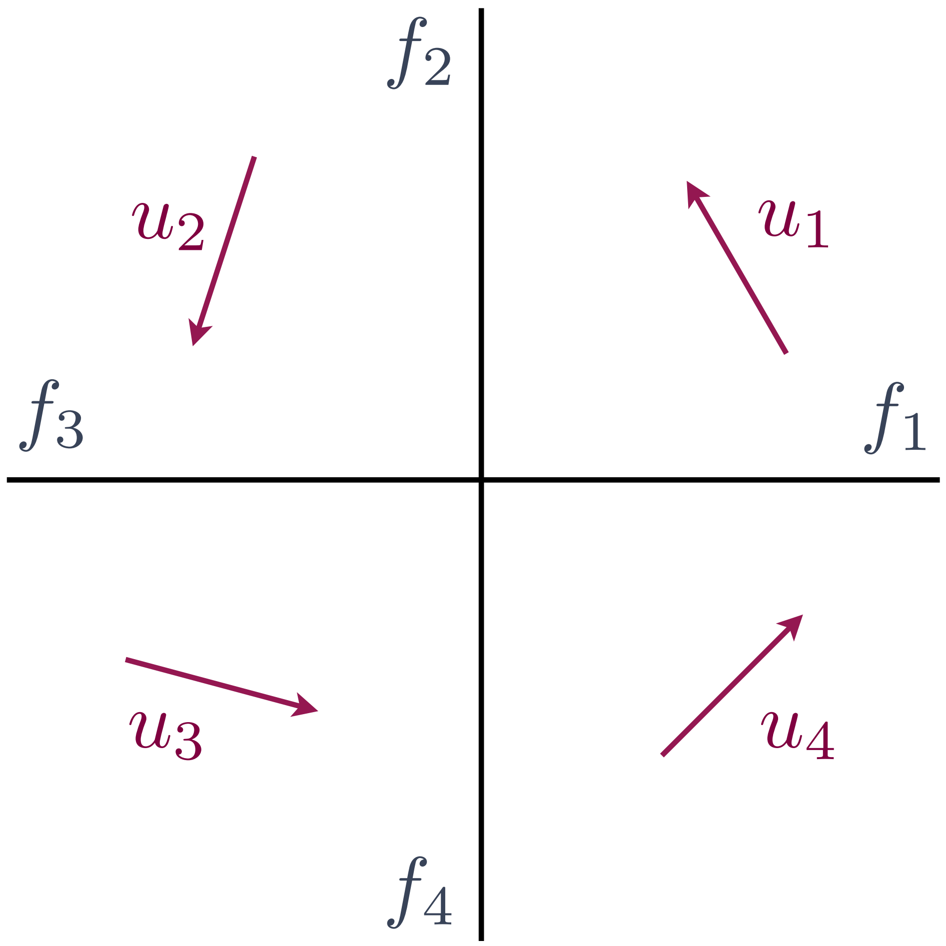


First CEGAR approach for stability verification of hybrid systems

CEGAR framework	Template based search
Systematically iterates over the abstract systems	Success depends crucially on the choice of the template
Returns a counterexample in the case that the abstraction fails	The current methods provide no insight into the reason for the failure, when a template fails to prove stability
The counterexample can be used to guide the choice of the next abstraction	No guidance regarding the choice of the next template

Quantitative Predicate Abstraction

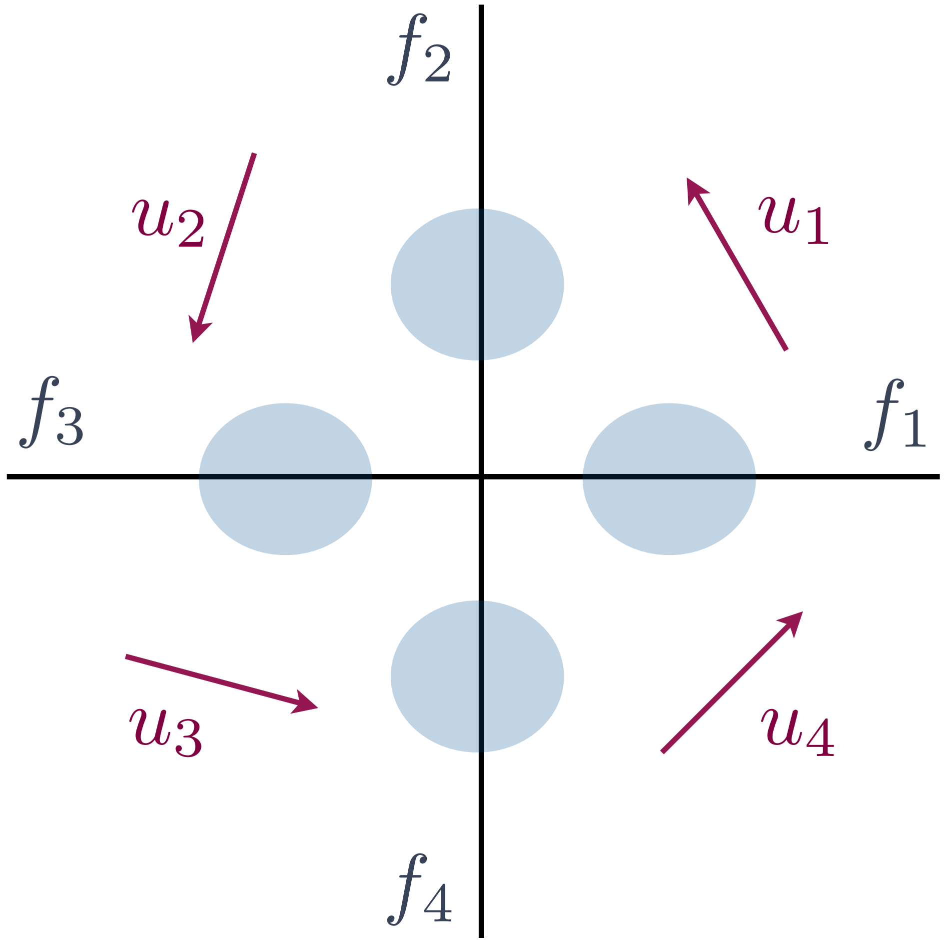
Quantitative Predicate Abstraction



Concrete system

Facets $\mathcal{F} = \{f_1, f_2, f_3, f_4\}$

Quantitative Predicate Abstraction



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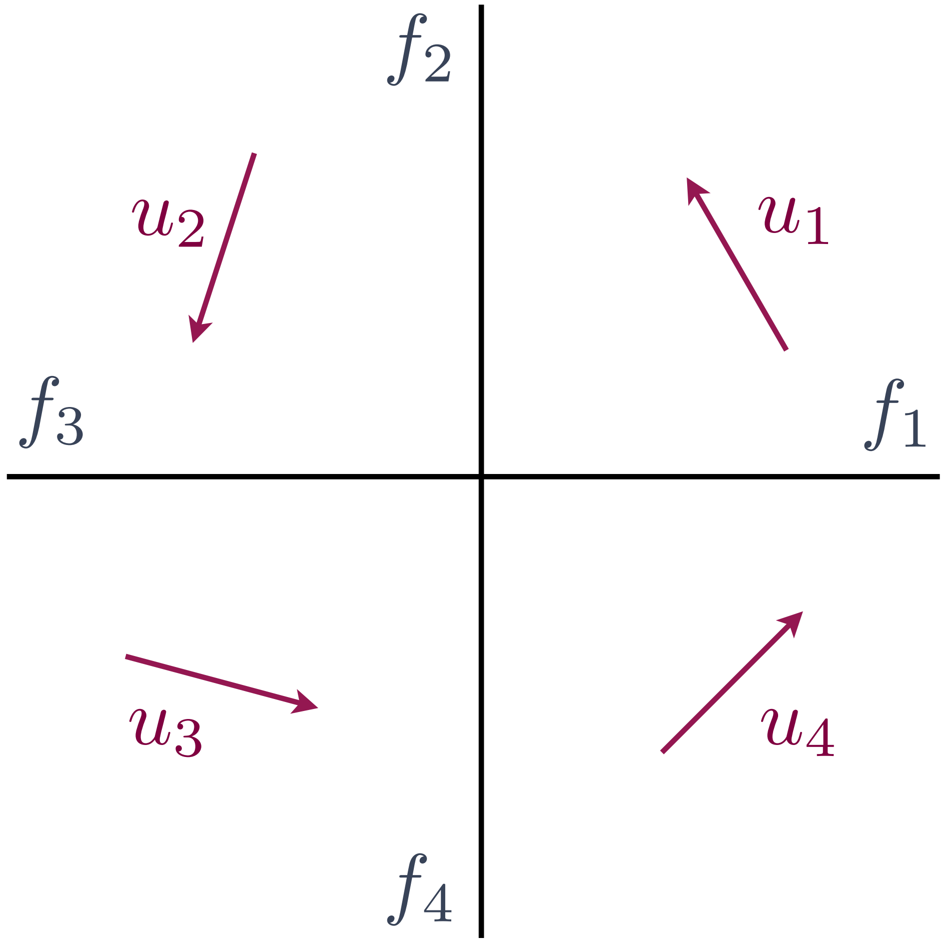

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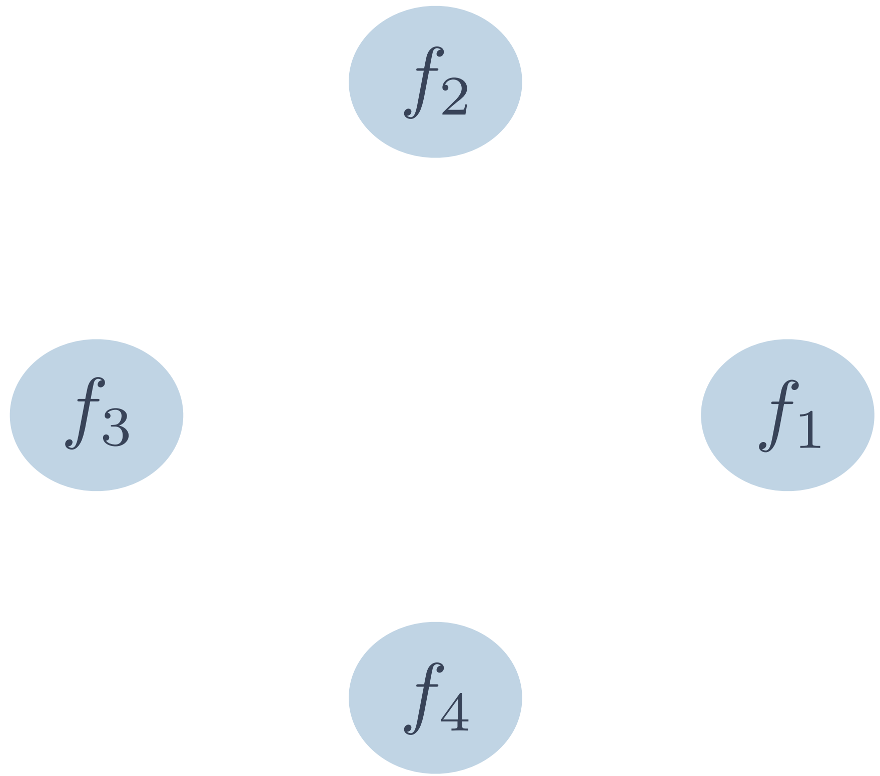
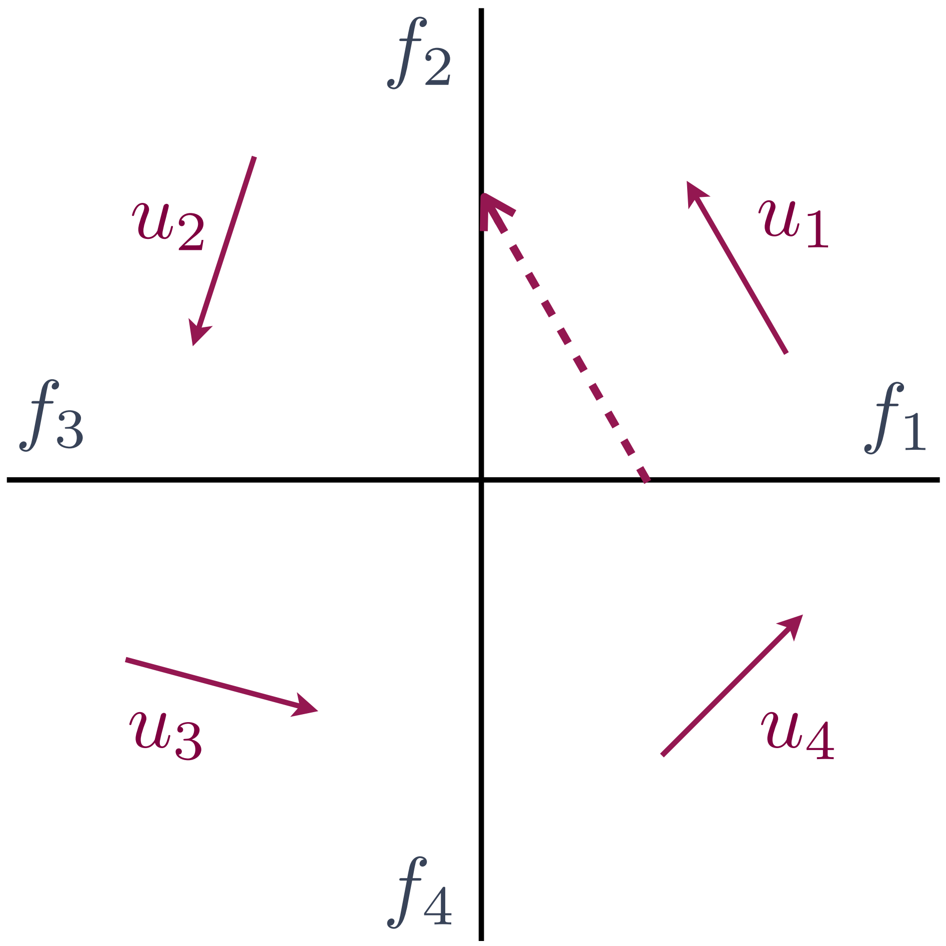


Concrete system

Abstract system

Facets $\mathcal{F} = \{f_1, f_2, f_3, f_4\}$

Quantitative Predicate Abstraction



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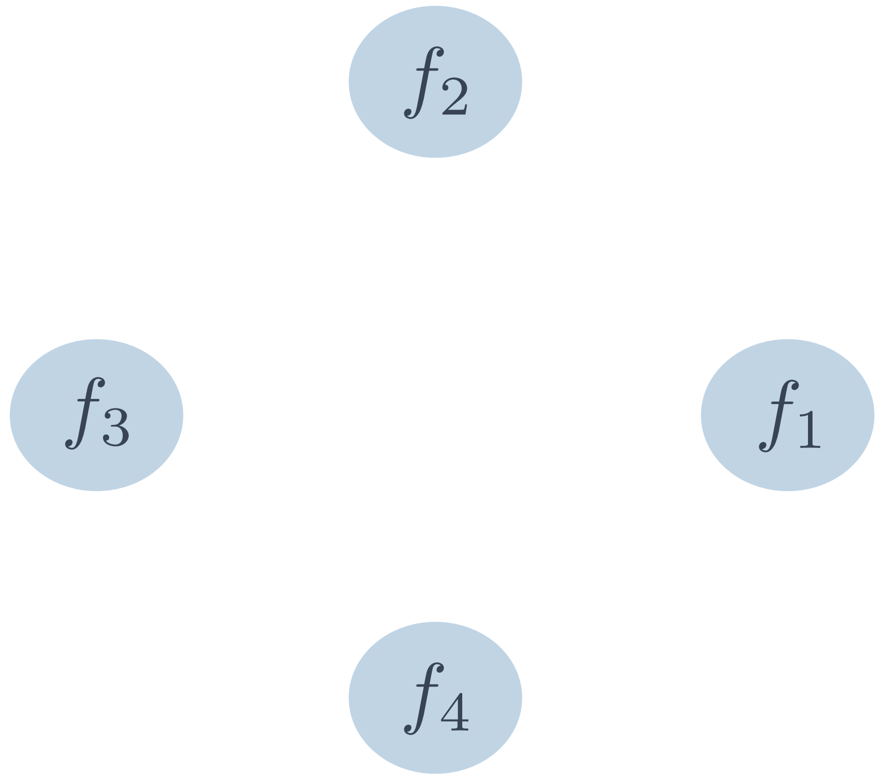
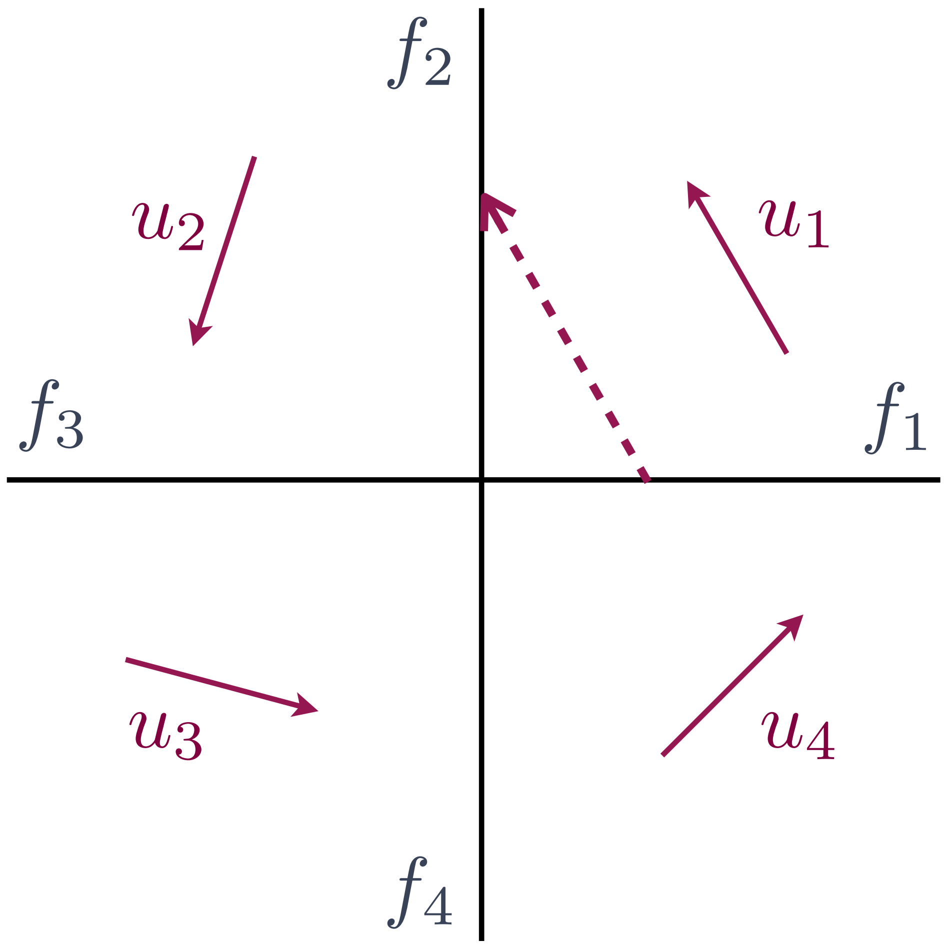


Abstract system

Quantitative Predicate Abstraction



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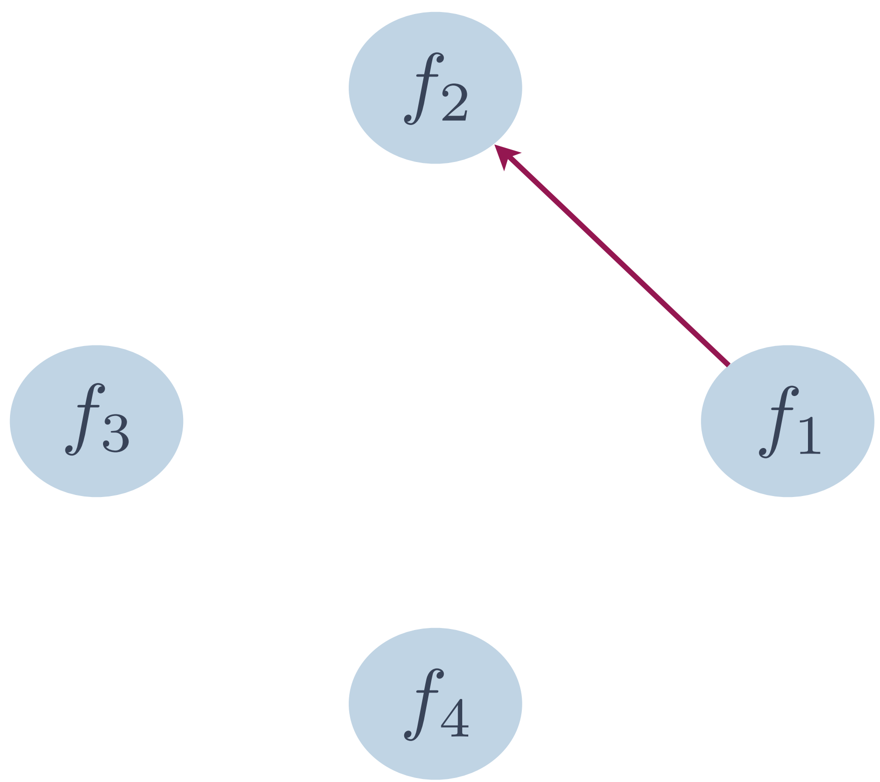


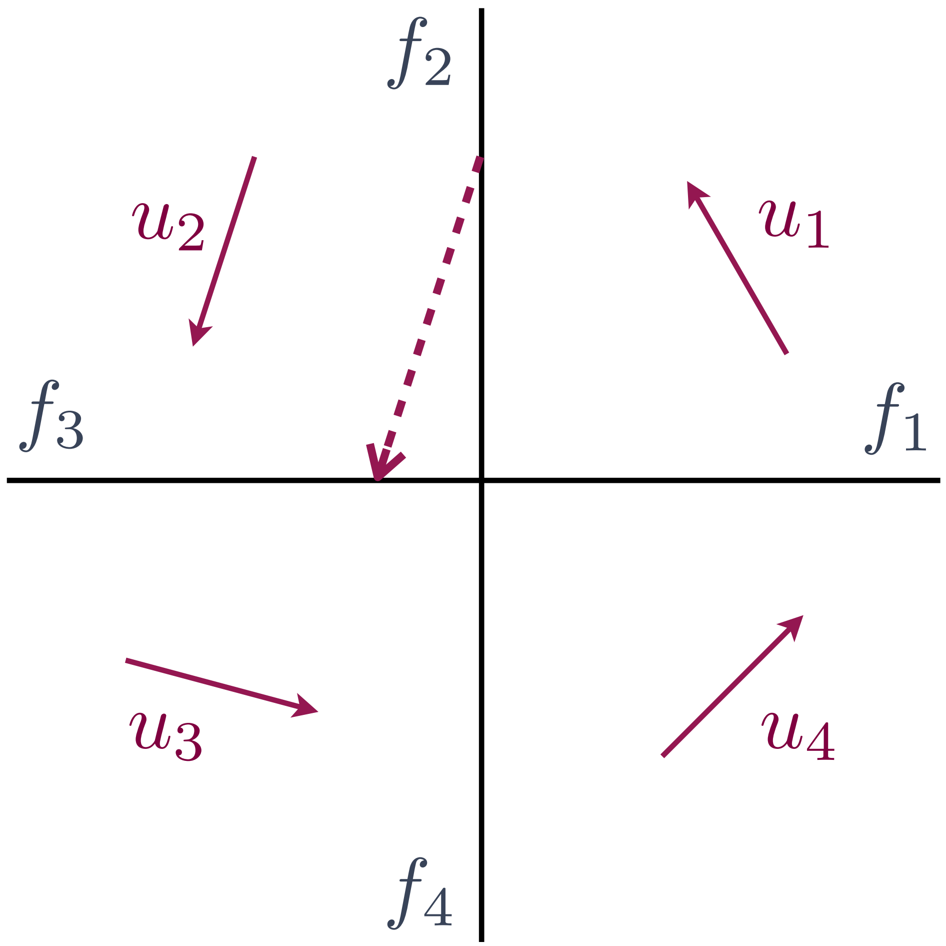
Concrete system

Abstract system

Facets $\mathcal{F} = \{f_1, f_2, f_3, f_4\}$

An edge between facets indicates the existence of an execution.

Quantitative Predicate Abstraction



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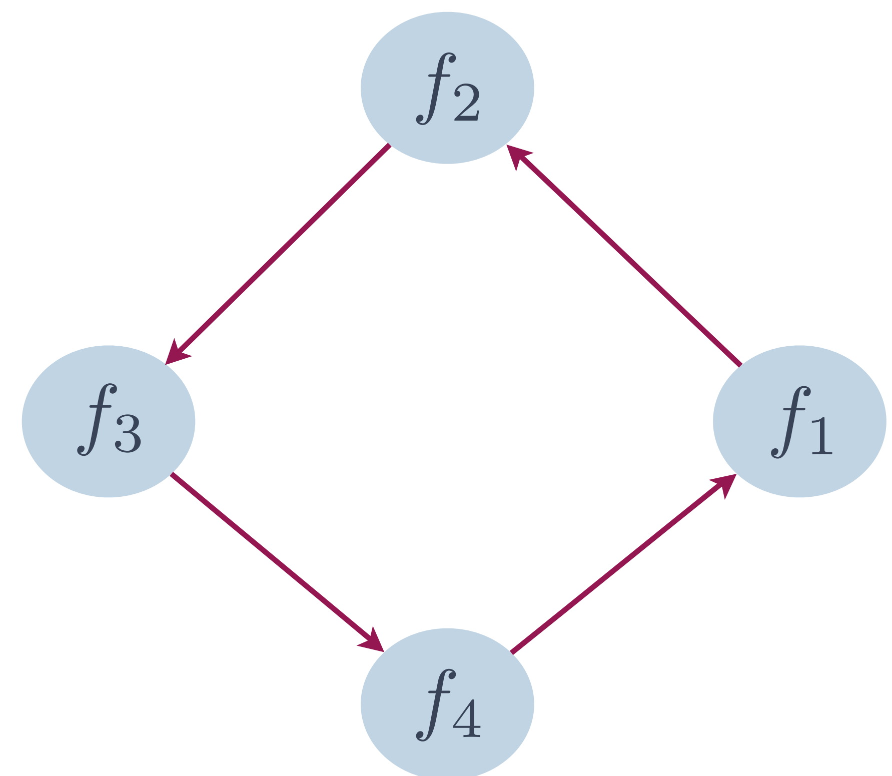

Concrete system

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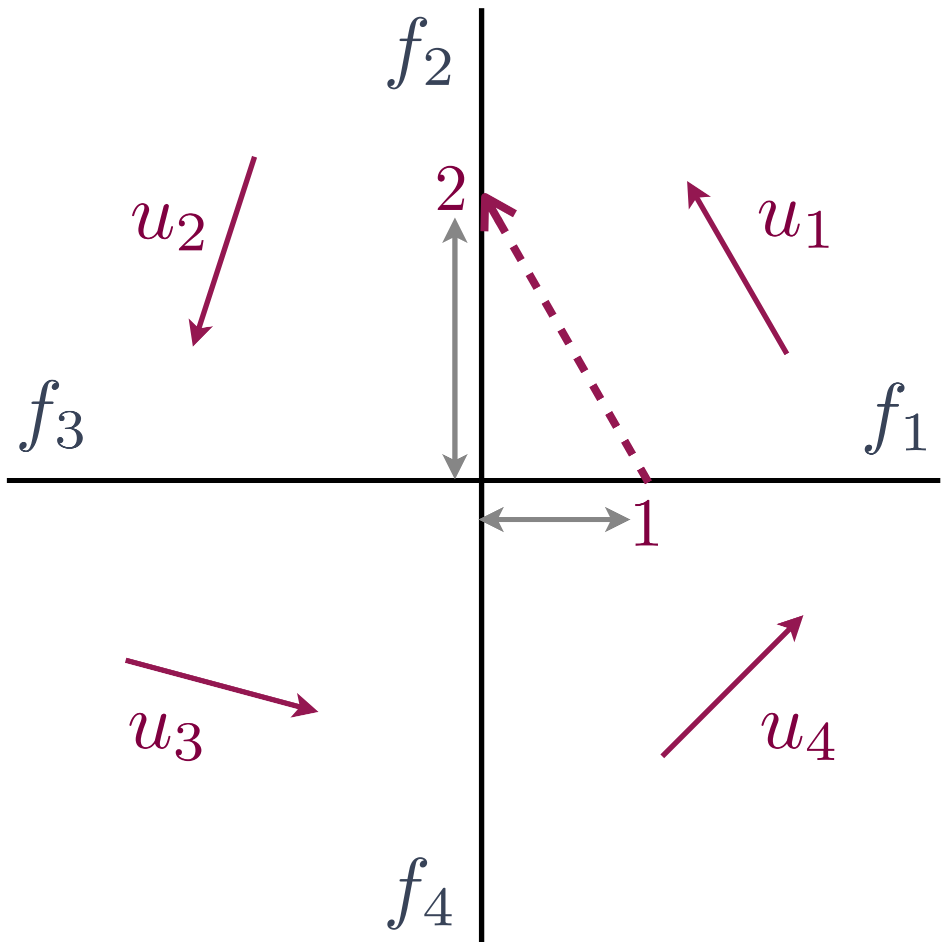
Abstract system

$\boxed{\textit{An edge between facets indicates the existence of an execution.}}$

Quantitative Predicate Abstraction



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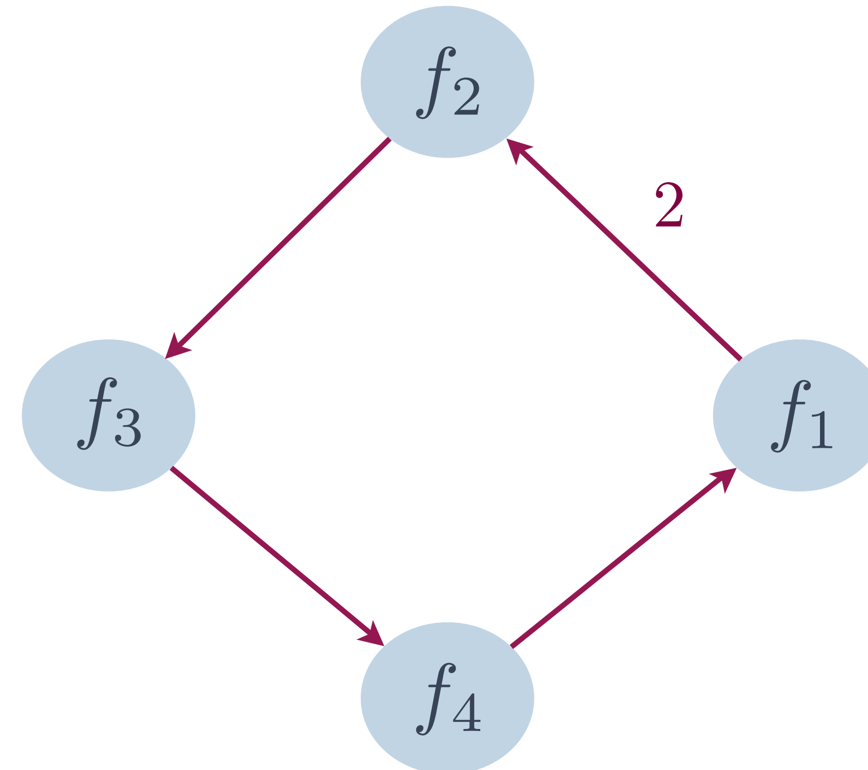


Concrete system

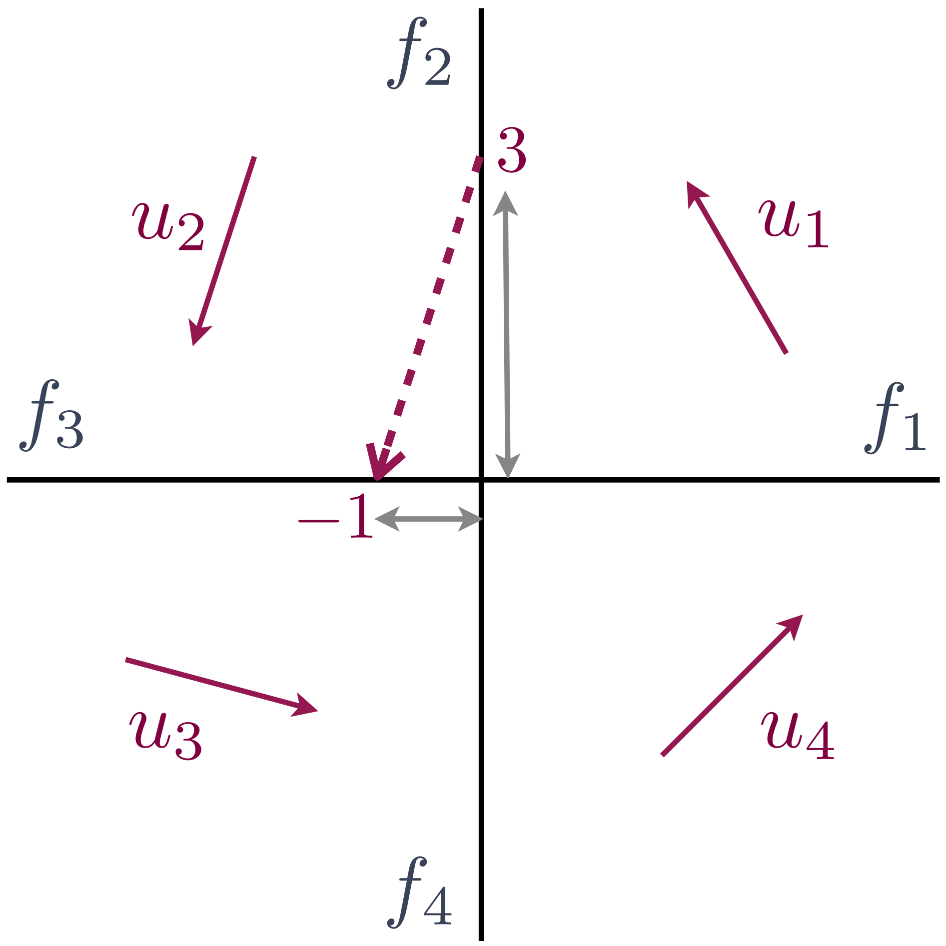
Abstract system

Facets $\mathcal{F} = \{f_1, f_2, f_3, f_4\}$

An edge between facets indicates the existence of an execution.

Weights capture information about distance to the equilibrium point along the executions.

Quantitative Predicate Abstraction



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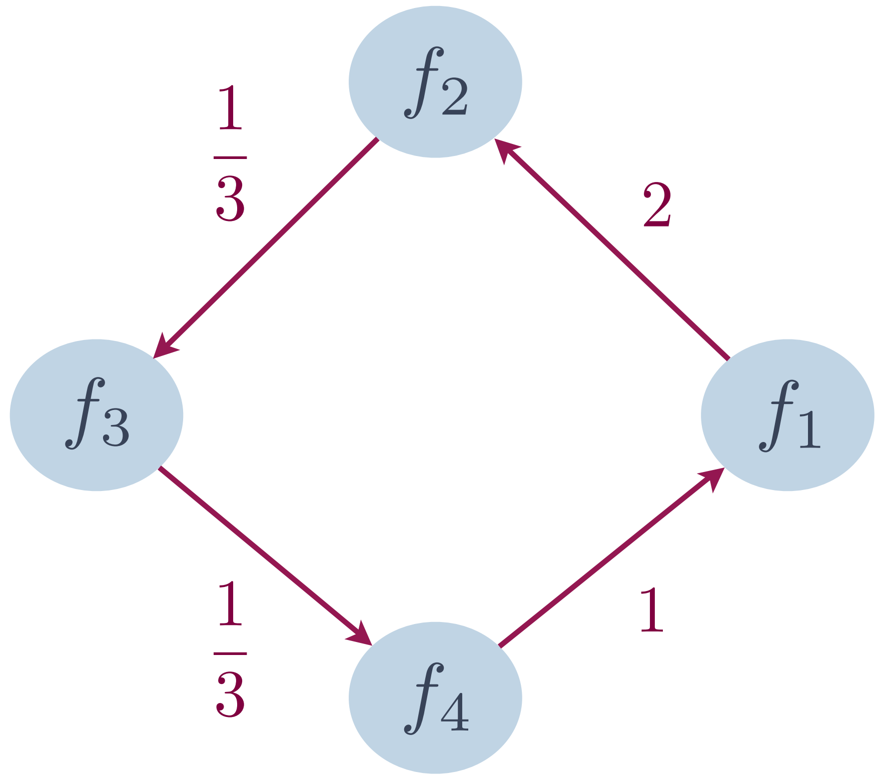

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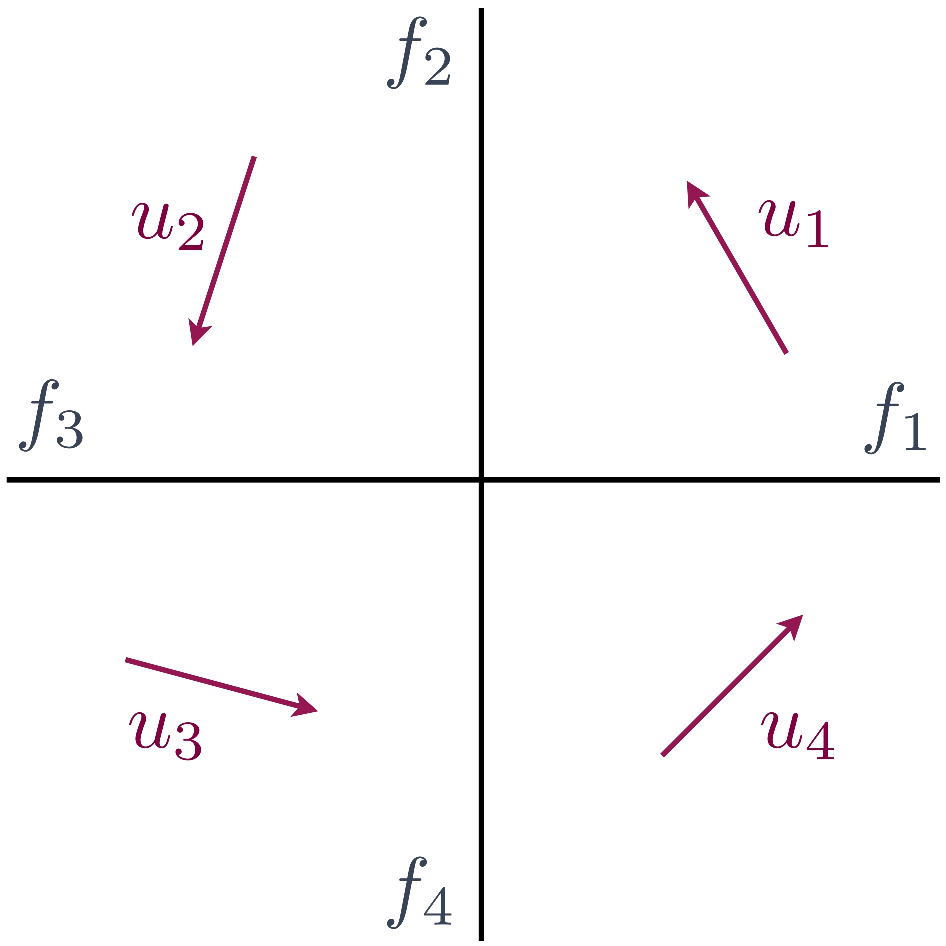

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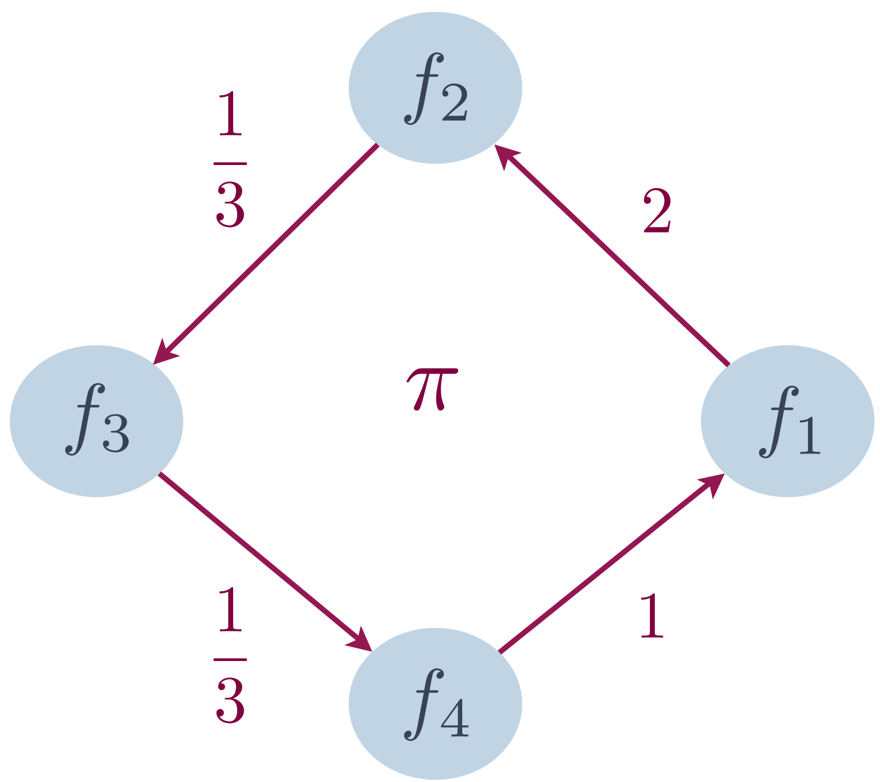


Concrete system

Facets $\mathcal{F} = \{f_1, f_2, f_3, f_4\}$

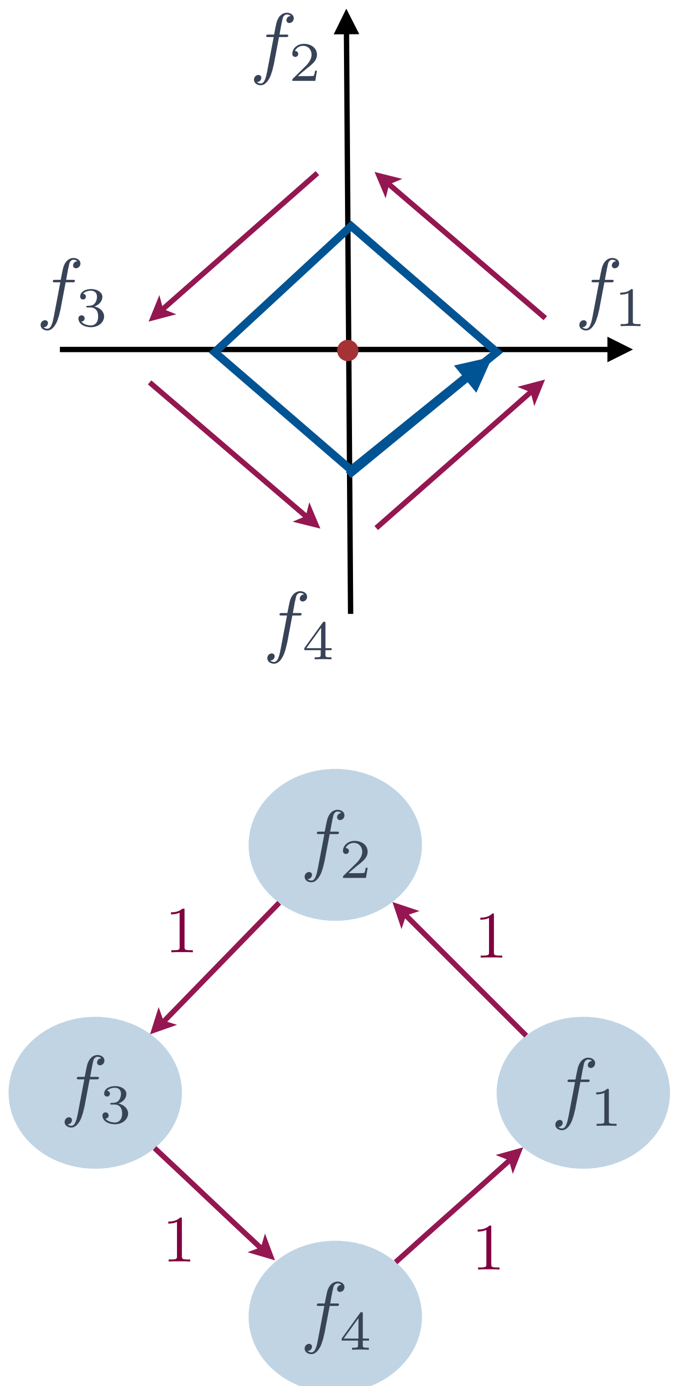
Abstract system

$$W(\pi) = 2 \cdot \frac{1}{3} \cdot \frac{1}{3} \cdot 1 = \frac{2}{9} < 1$$

An edge between facets indicates the existence of an execution.

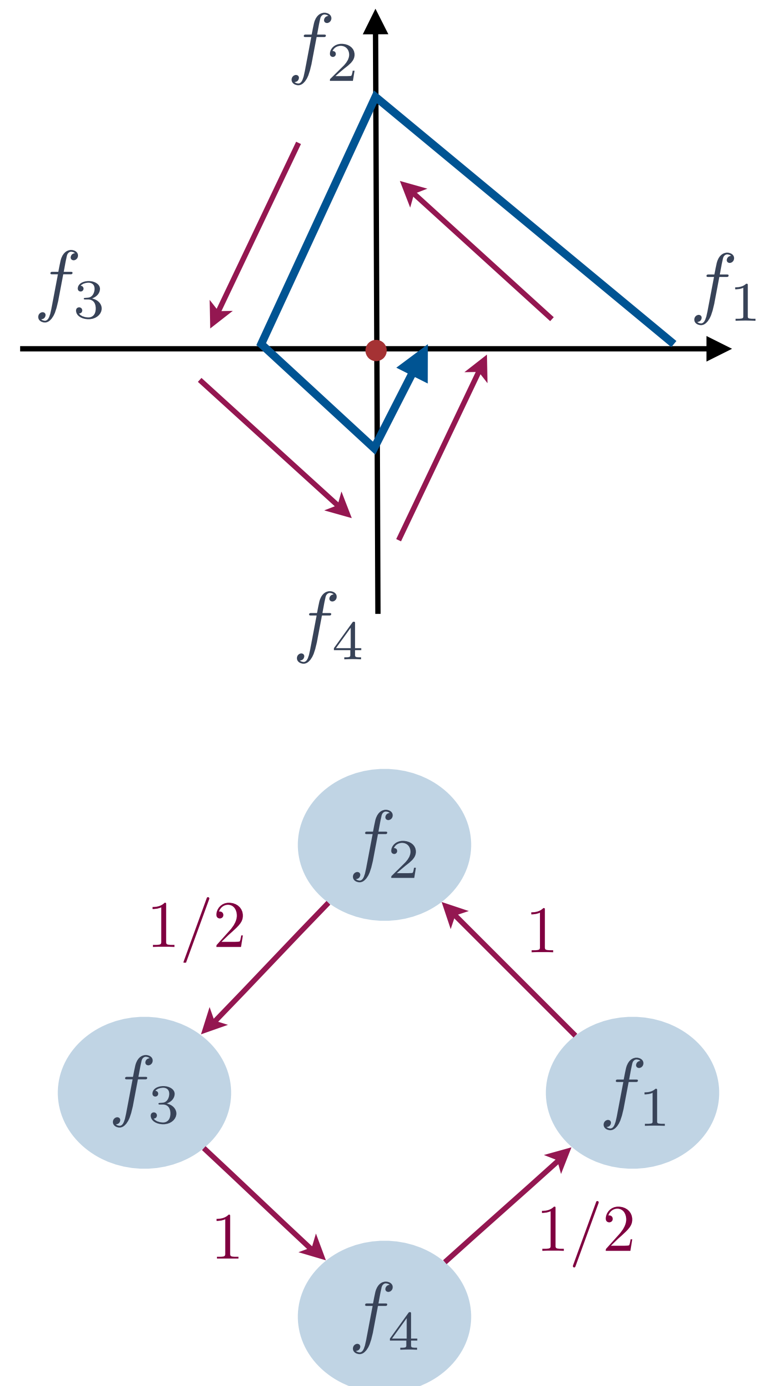
Weights capture information about distance to the equilibrium point along the executions.

Quantitative Predicate Abstraction - samples



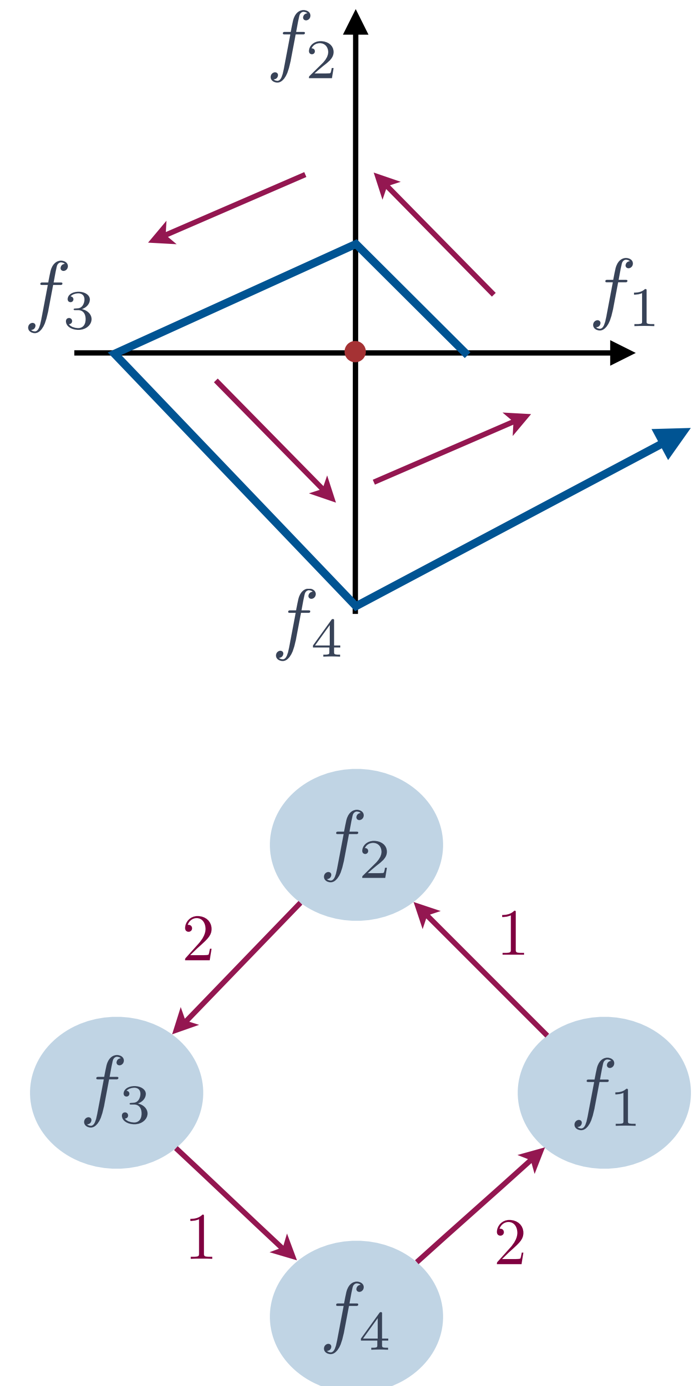
Product of edge weights = 1

Lyapunov Stable



Product of edge weights = 1/4

Asymptotically Stable



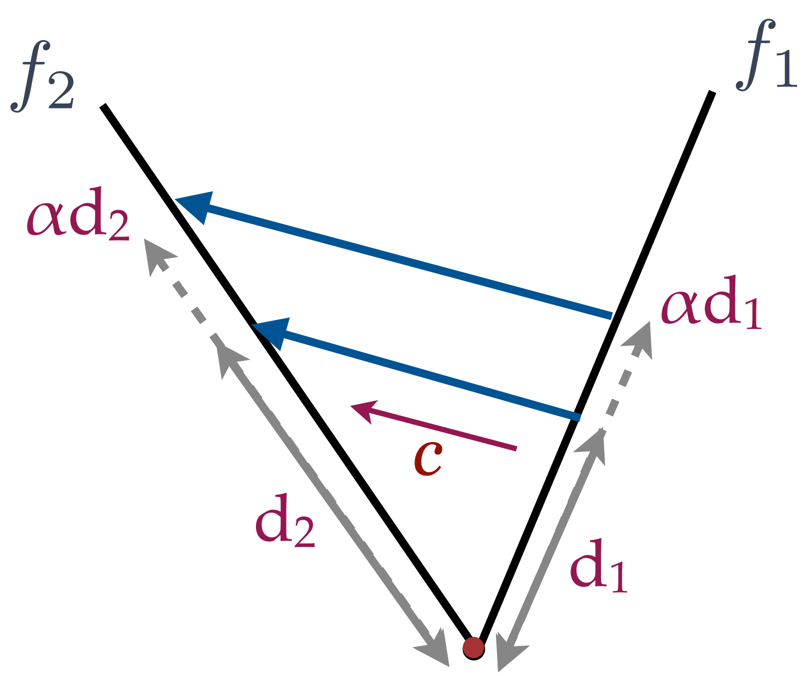
Product of edge weights = 4

Unstable

Weight computation

Constant dynamics $\dot{x} = c$

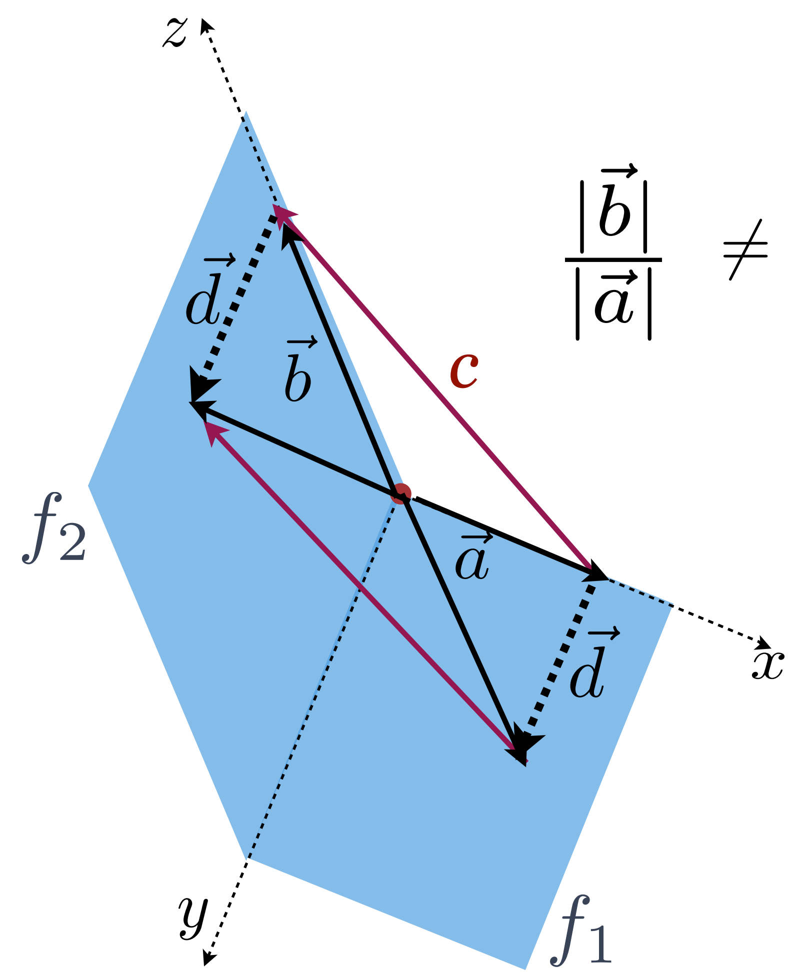
<u>2 dimension</u>



Weight

$$\boxed{\frac{|d_2|}{|d_1|} = \frac{|\alpha d_2|}{|\alpha d_1|}}$$

<u>Higher dimensions</u>



$$\frac{|\vec{b}|}{|\vec{a}|} \neq \frac{|\vec{b}+\vec{d}|}{|\vec{a}+\vec{d}|}$$

Weight (LP problems)

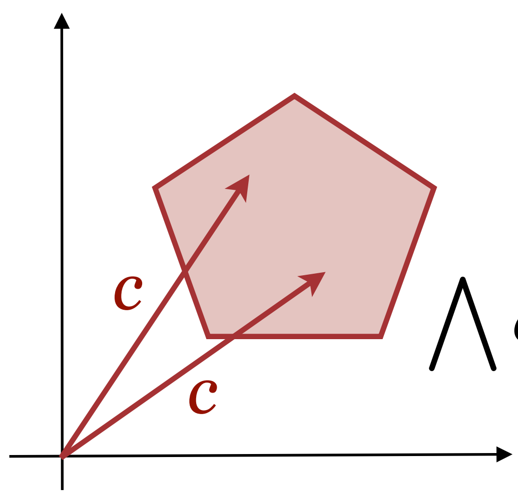
$$\sup \frac{|v_2|}{|v_1|}$$

$$t \geqslant 0, v_1 \in f_1, v_2 \in f_2, v_2 = v_1 + ct$$

Weight computation

Polyhedral inclusion dynamics $\dot{x} \in P$

P is a polyhedral set



$$\bigwedge a_i \cdot x \leq b_i$$

Weight (LP problems)

$$\sup \frac{|v_2|}{|v_1|}$$

$$\bigwedge a_i \cdot (v_2 - v_1) \leqslant b_i t$$

$$t \geqslant 0, v_1 \in f_1, v_2 \in f_2, \cancel{v_2 = v_1 + ct, \bigwedge a_i \cdot c \leqslant b_i}$$

Weight computation

Linear dynamics $\dot{x} = Ax$

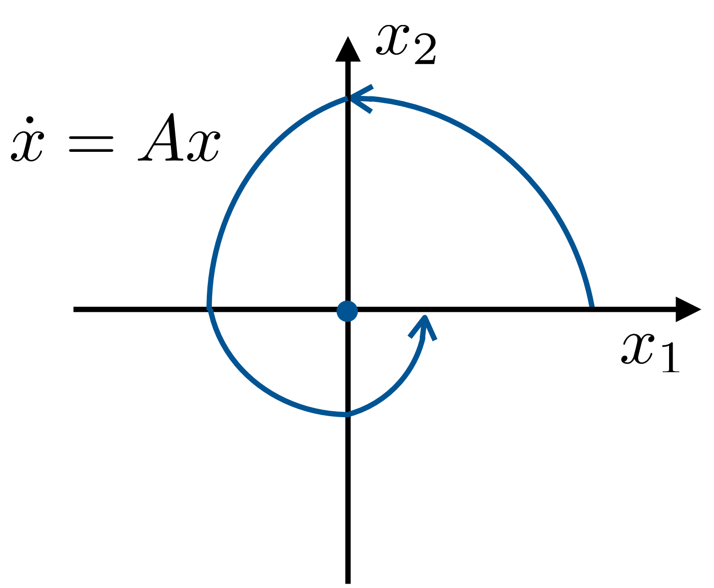
Weight

$$\boxed{\begin{array}{c} \sup \dfrac{|v_2|}{|v_1|} \\[2ex] t \geqslant 0, v_1 \in f_1, v_2 \in f_2, v_2 = v_1 e^{At} \end{array}}$$

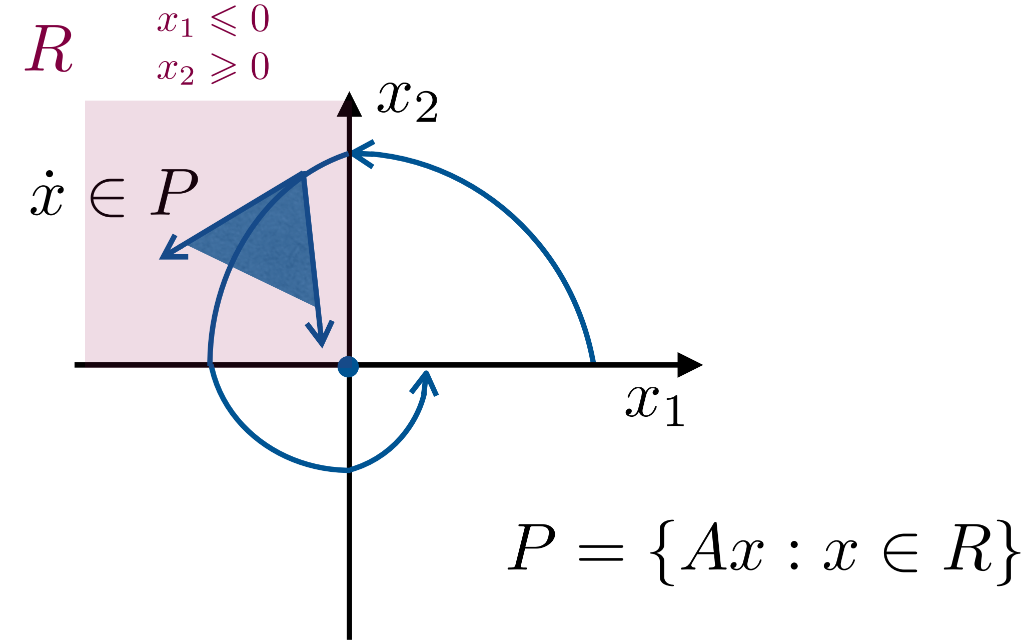
- Solution is an exponential function
- Need a representation on which optimization can be performed
- Approximation methods [Girard et al., Frehse et al.]

Hybridization

Hybridization and soundness



Linear hybrid system



Polyhedral hybrid system

Theorem - Hybridization

If the hybridized polyhedral hybrid system is Lyapunov (asymptotically) stable then the original linear hybrid system is Lyapunov (asymptotically) stable.

Hybridization for stability analysis of switched linear systems. HSCC'16

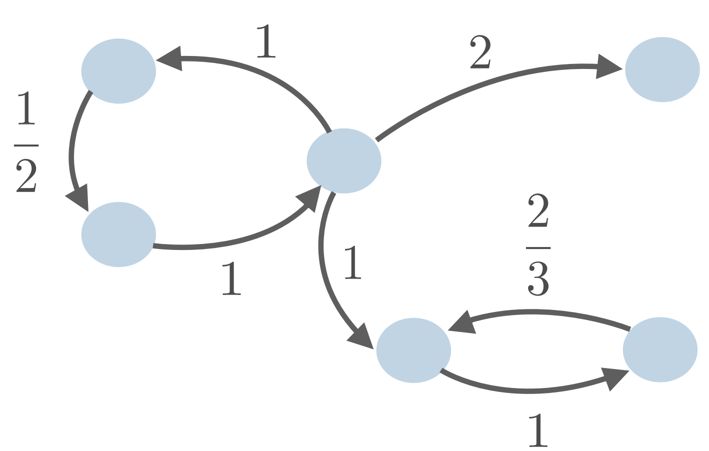
Soundness of Quantitative Predicate Abstraction

Theorem - Model-checking

A polyhedral hybrid system is Lyapunov stable if

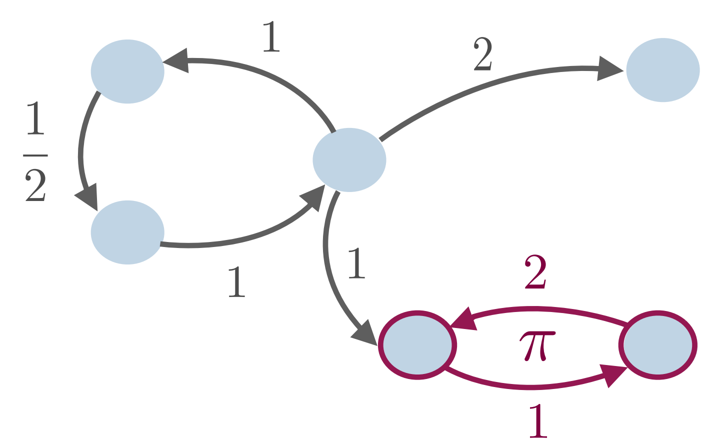
- the abstract weighted graph has no edges with infinite weights, and
- no cycles with product of edge weights greater than 1

Abstract system



Every cycle has weight smaller than 1 => Concrete system is stable => Stop

Abstract system



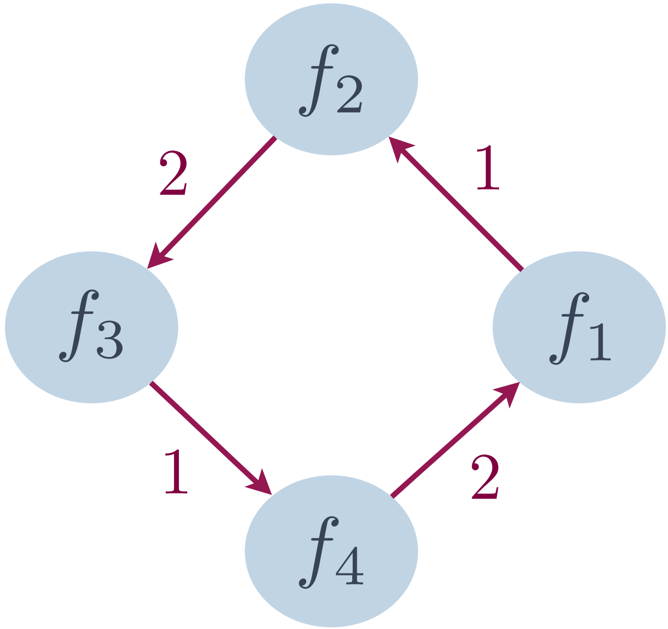
There is a cycle, π, with weight greater than 1 => π is an abstract counterexample => Validation

Abstraction based model-checking of stability of hybrid systems. CAV'13

Foundations of Quantitative Predicate Abstraction for Stability Analysis of Hybrid Systems. VMCAI'15

Counterexample

- Model-checking of the abstract system returns an abstract counterexample if the abstract system fails to establish stability.



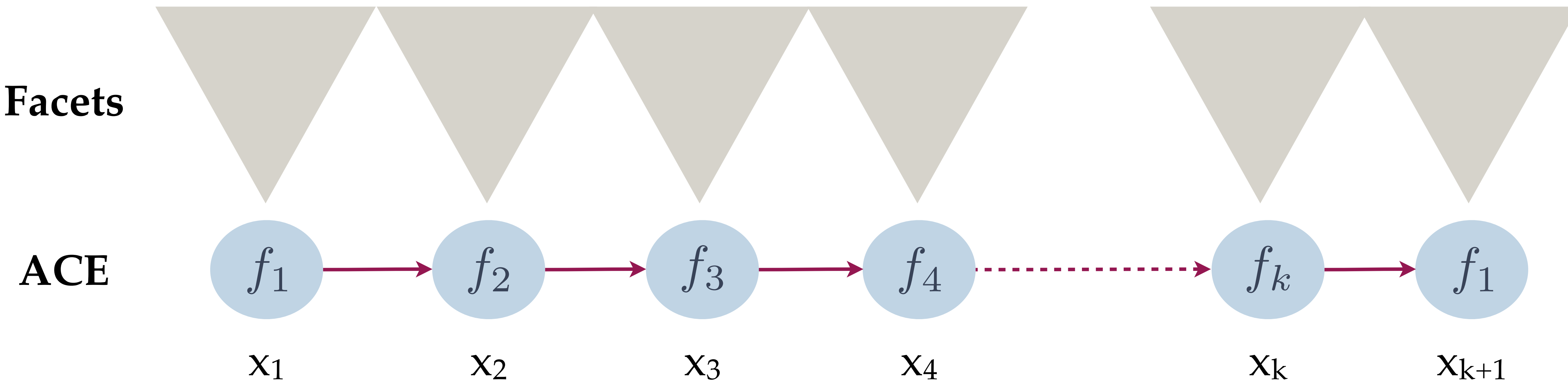
Abstract Counterexample (ACE):
A cycle with product of edge weights greater than 1

- **Spurious ACE:** If there exist no infinite execution **(concrete)** of the system which *follows* the edges and weights of the cycle (and diverges)
- **Validation:** Checking if the ACE is spurious.

Validation is not a bounded model-checking problem!
Requires checking for an infinite execution instead of a finite execution.

Validation

Validation



Theorem - Validation

A counterexample $f_1 \longrightarrow f_2 \longrightarrow f_3 \longrightarrow \ldots \longrightarrow f_1$ is valid

$$\Longleftrightarrow$$

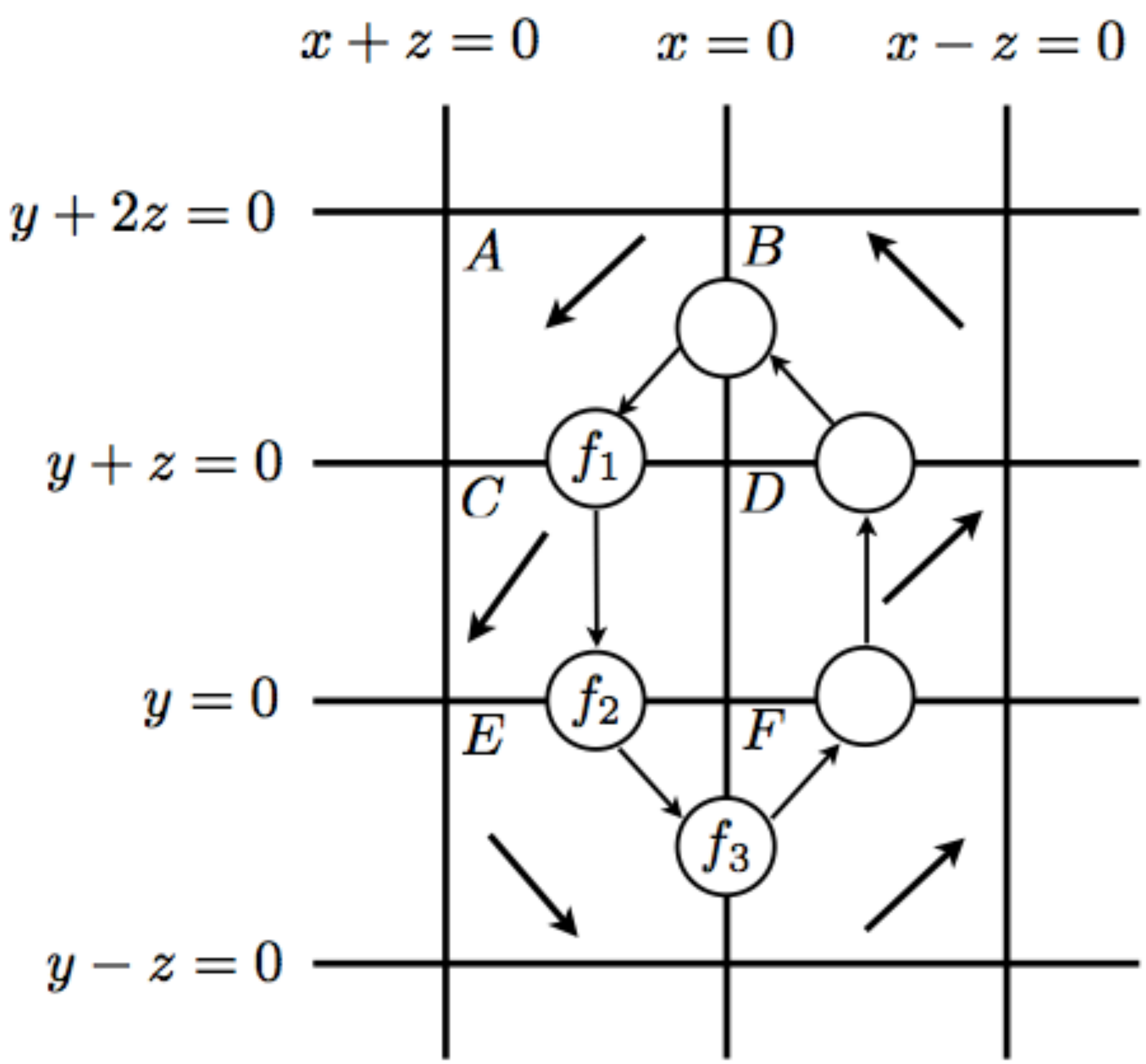
$$\exists\, \alpha > 1,\ \exists\, x_1 \in f_1, \ldots, x_k \in f_k,\ x_{k+1} \in f_1$$

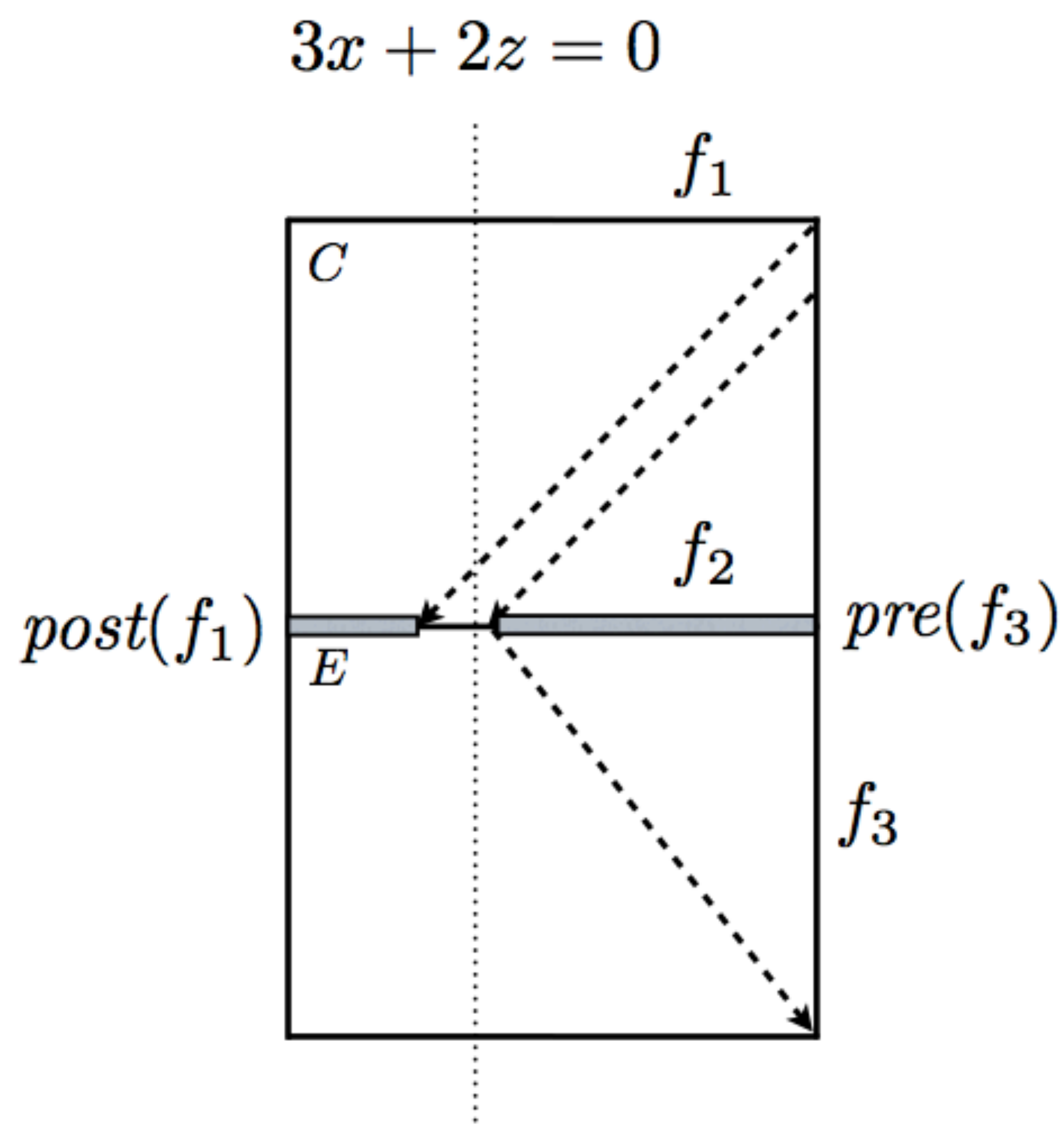
$$x_1 \longrightarrow x_2 \longrightarrow x_3 \longrightarrow \ldots \longrightarrow x_k \longrightarrow x_{k+1},\ x_{k+1} = \alpha x_1$$

Existence of an infinite concrete counterexample is equivalent to the existence of a finite execution along the cycle with certain properties, which can be encoded as an ***SMT formula****.*

Refinement

Refinement

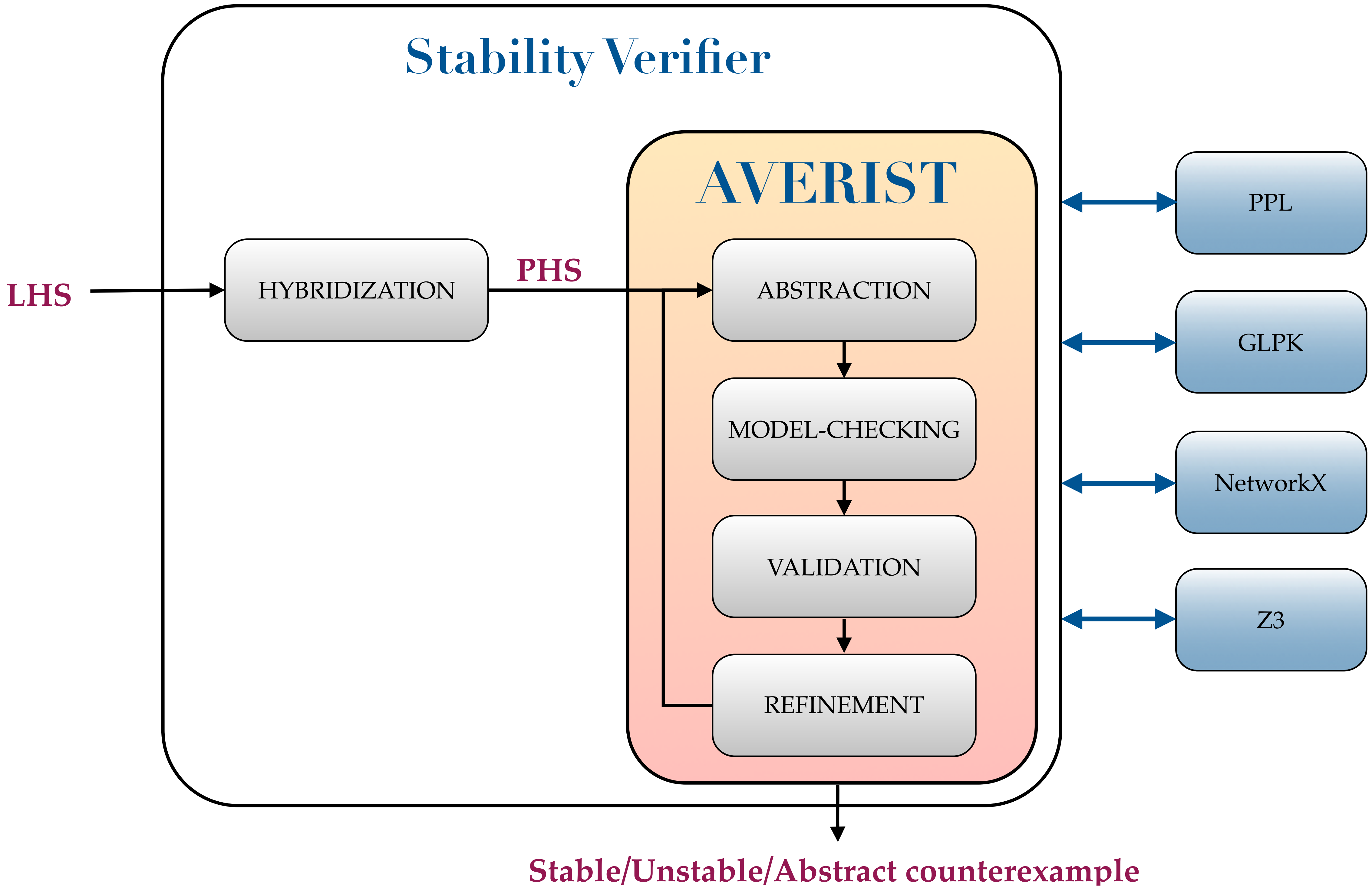




Counterexample guided abstraction refinement for stability analysis. CAV'16

Software tool

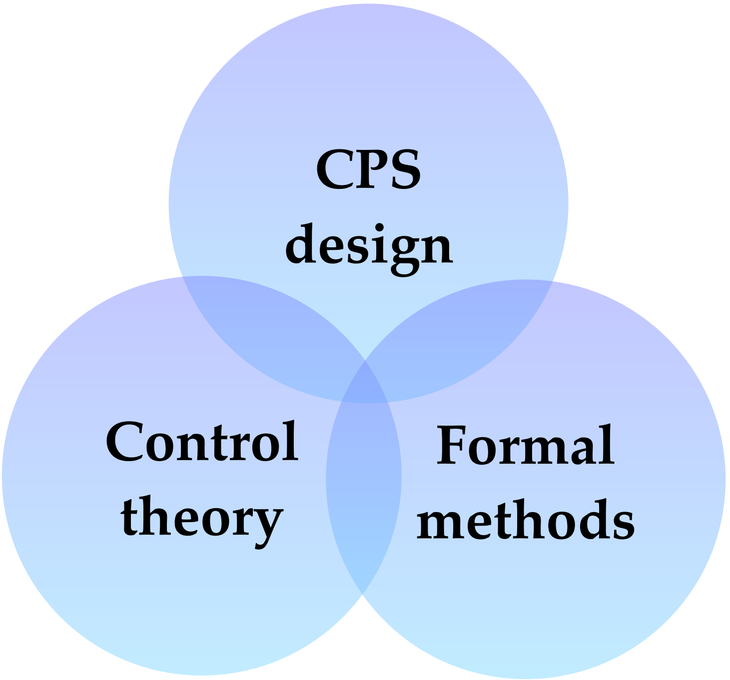
AVERIST flowchart and software dependencies



http://software.imdea.org/projects/averist/index.html

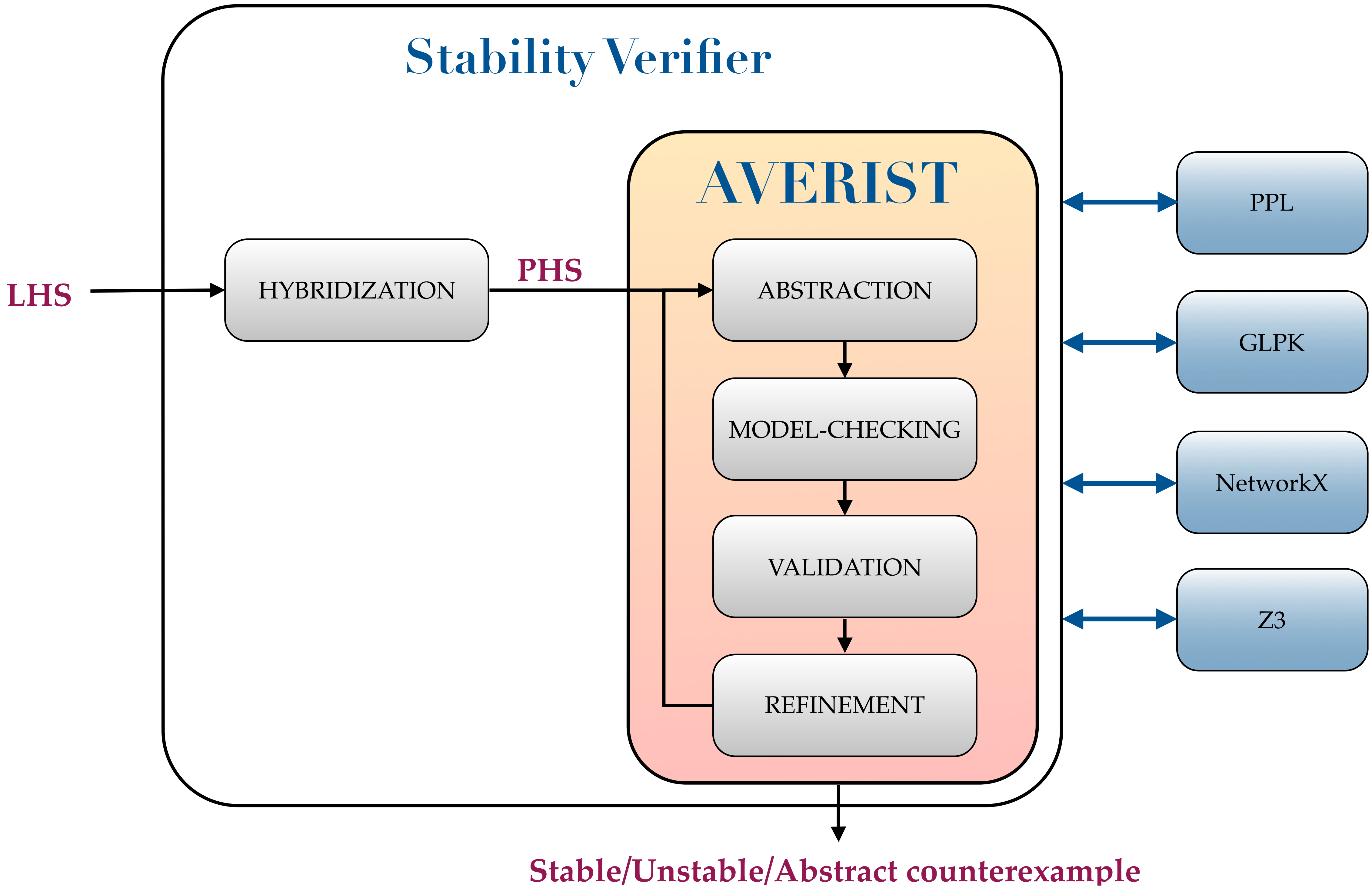
Conclusion

Summary



- Development of a novel **CEGAR approach**, based on abstraction and model-checking techniques
- **Automatic** process for **linear** and **polyhedral hybrid systems**
- **Framework extendable** to more complex class of hybrid systems
- Techniques implemented in **AVERIST** provide promising results
- Application to an **automatic gearbox**

Questions?



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